5. Hydrostatic Balance and Geopotential

The hydrostatic equation

Except for motions on rather small horizontal scales, the atmosphere is in a state of close hydrostatic balance. That is, the net pressure force acting on a small element of air in a vertical column is equal to the weight of air in the element (Fig. 5.1). Mathematically, this force balance can be expressed by the differential relation

$$p(z+\Delta z)\,A \;+\; \rho\, g\, A\Delta z \;=\; p(z)\,A,$$

or, as $\Delta z \to 0$,

$$\frac{dp}{dz} = -g\rho \tag{5.1}$$

This is called the hydrostatic equation. The negative sign arises because pressure decreases with height.

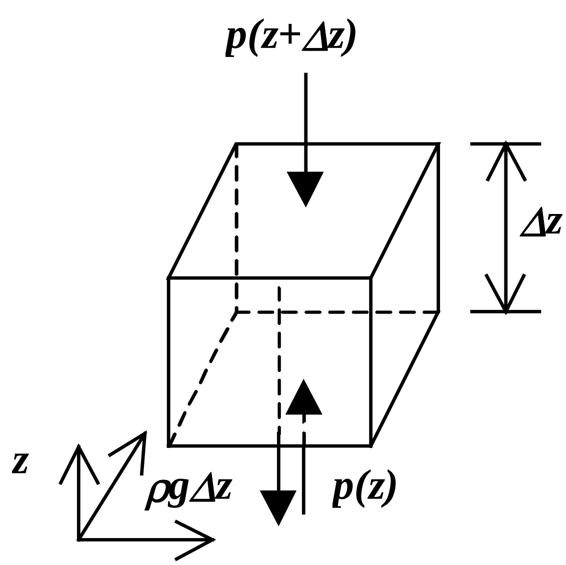

Figure 5.1. Hydrostatic balance.

Given the variation of density with height, $\rho(z)$, Eq. (5.1) may be integrated with respect to *z*, making use of the conditions that $p(z) \to 0$ as $z \to \infty$. Therefore

$$p(z) = \int_z^{\infty} g\,\rho(z)\,dz. \tag{5.2}$$

This shows that the pressure at height z is equal to the weight of the air in vertical column of unit area lying above that level.

Alternatively, using the perfect gas equation $p = \rho R_d T$ to eliminate ρ in preference to T, Eq. (5.1) becomes

$$\frac{1}{p}\frac{dp}{dz} = -\frac{g}{R_d T}.$$

This equation integrates at once to give

$$p(z) = p(0)\exp\left[-\frac{g}{R_d}\int_0^z \frac{dz}{T(z)}\right]. \tag{5.3}$$

Equation (5.3) gives the pressure as a function of height in terms of the vertical distribution of temperature $T(z)$ and the surface pressure $p(0)$.

Suppose that the temperature T is constant and equal to $\overline{T}$ in some layer, i.e, the layer is *isothermal*. Alternatively, $\overline{T}$ could be the average temperature in a layer. Then Eq. (5.3) becomes

$$p(z) \;=\; p(0)\exp\left(-\frac{g\,z}{R_d\overline{T}}\right) \;=\; p(0)\exp\left(-\frac{z}{H_s}\right), \tag{5.4}$$

where $H_s = R_d\overline{T}/g$ is known as the scale height. The scale height represents an *e*-folding height scale. In other words, the pressure decreases by a factor of 1/e over this height. For a temperature of 288 K near the Earth's surface $H_s \approx 8.5\,km$. From the perfect gas law, the variation of density with height is given by

$$\rho(z) \;=\; \rho(0)\exp\left(-\frac{z}{H_s}\right),$$

where $\rho(0)$ is the density at z = 0. Thus, both the pressure and temperature decrease exponentially with height. The atmosphere is compressible under its own weight; lower layers are compressed more than upper layers.

The typical vertical profile of temperature, density and pressure up to a height of about 100 km are shown in Fig. 5.2. The different layers of the atmosphere are defined by the different variations in the temperature with height.

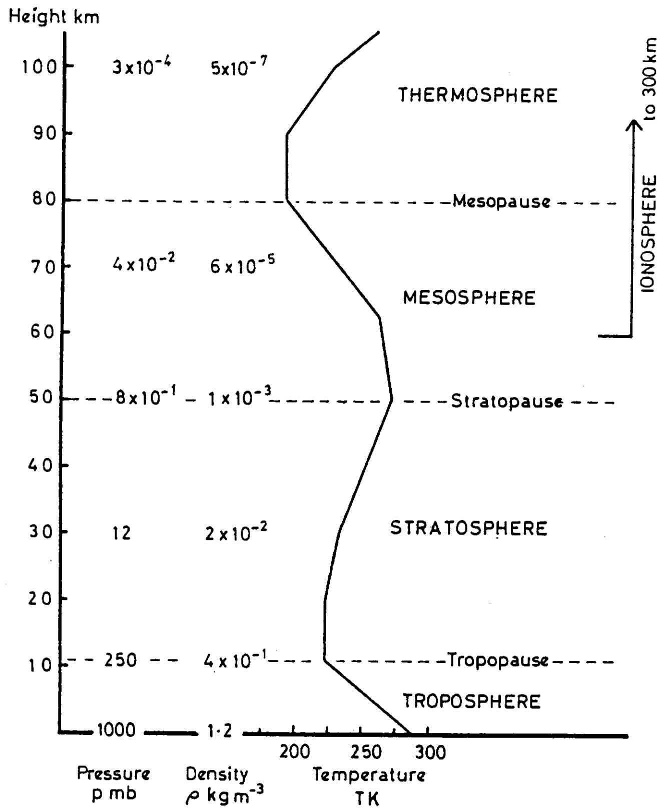

Figure 5.2. Typical vertical profiles of pressure, temperature and density.

Pressure as a vertical coordinate

The use of p as the vertical coordinate is a practical alternative to using z because of the strong hydrostatic relationship that exists in the vertical under many circumstances of interest, and because of the strong functional relationship between pressure and height (i.e., exponential). Figure 5.3 shows how closely height and log(pressure) are related. It is often easier to measure pressure than the measure height, for example, radiosondes transmit their pressure and temperature from which the height is calculated.

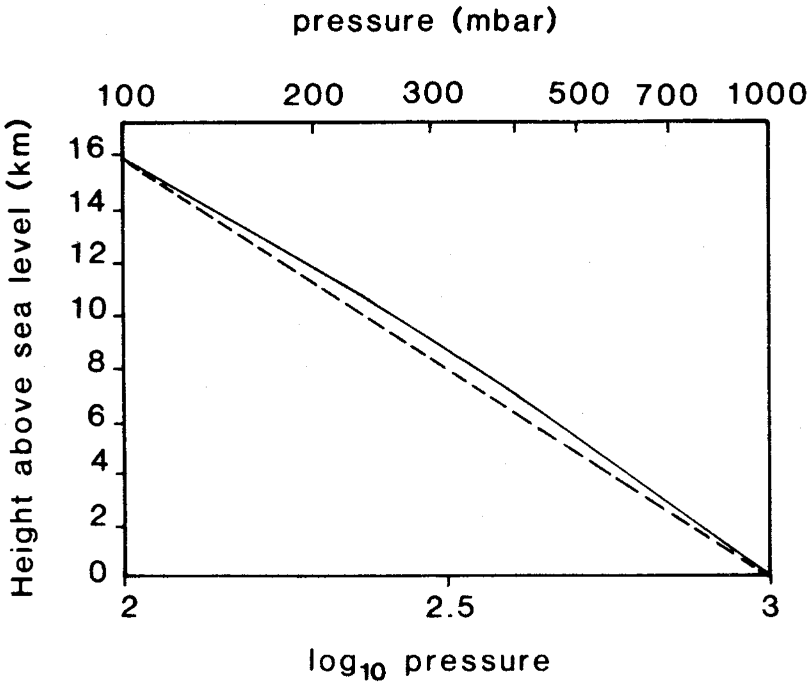

Figure 5.3 Pressure plotted against height for a typical sounding (UK in winter), from McIlveen 1992. The dashed line shows a straight line.

Geopotential

The *geopotential* Φ at any point in the atmosphere in defined as the work that must be done against the Earth's gravitational field in order to raise a mass of 1 *kg* from sea level to that point. Expressed another way, Φ is the gravitational potential for a unit mass. It has units $J\ kg^{-1}$, or $m^2 s^{-2}$. The force (in newtons) acting on 1 *kg* at height z above sea level is numerically equal to g. The work done in raising $1\ kg$ from z to $z + dz$ is $d\Phi = g\,dz$. Since the depth of the atmosphere is so small in comparison to the radius of the Earth, the acceleration due to gravity is usually taken to be constant in (9.8 ms^{-1}) meteorological problems. In this case, $\Phi = g\,z$. Otherwise, $\Phi(z) = \int_0^z g(z)dz$, assuming $\Phi(0) = 0$.

The *geopotential height* is defined as $Z = \Phi/9.8$. Of course, if g is taken to be a constant, then z = Z.

Because of the equation of state,

$$p = \rho R_d T\ , \qquad (5.5)$$

it is necessary to measure only two of the three quantities p, ρ, and T. Since p and T are relatively easy to measure, the density, when required, is obtained from Eq. (5.5).

Wherever possible, it is convenient to eliminate ρ from the equations. For example, using Eq. (5.5) the hydrostatic equation can be written

$$\frac{dp}{dz} = -\frac{p\,g}{R_d\,T} \quad . \tag{5.6}$$

Rearranging this, and using the relation $g\,dz = d\Phi$ gives

$$d\Phi = -R_d T \frac{dp}{p} \ ,$$

whereupon

$$\Phi_2 - \Phi_1 = -R_d \int_{p_1}^{p_2} T \frac{dp}{p} \ ,$$

or

$$z_2 - z_1 = \frac{R_d}{g} \int_{p_2}^{p_1} T \ \frac{dp}{p} . \tag{5.7}$$

Thus, the (geopotential) heights can be determined at a particular place on the Earth by measuring the variation of temperature as a function of pressure. Such measurements are made by radiosonde soundings of the atmosphere.

Upper level synoptic charts in practical use are drawn on isobaric surfaces and the principal isoline on these charts are the geopotential height contours. Since these isobaric surfaces are almost horizontal to a close approximation, the component of air motion in this surface is to within measurable accuracy equal to the horizontal wind speed.

For a dry, isothermal atmosphere $T = \text{constant}$, and Eq. (5.7) may be integrated to give

$$z_2 - z_1 = (RT/g)\ln(p_1/p_2),$$

In the troposphere, the temperature varies with height, but the formula Eq. (5.7) can still be used if we define T to be $\overline{T}$, the mean temperature with respect to $\ln p$, i.e.,

$$\overline{T} \quad = \quad \frac{\int_{\ln p_2}^{\ln p_1} T\, d\,(\ln p)}{\int_{\ln p_2}^{\ln p_1} d\,(\ln p)} \quad = \quad \frac{\int_{p_2}^{p_1} T\, \frac{dp}{p}}{\ln\left(\frac{p_1}{p_2}\right)} \ . \tag{5.8}$$

Thickness and height of isobaric surfaces

The difference in (geopotential) height $z_2 - z_1$ between any two levels in the atmosphere is called the *thickness* of the intervening layer. The foregoing theory shows that the thickness between two isobaric surfaces p_1 and p_2 is proportional to the mean temperature between these surfaces. Thus, if $\overline{T}$ increases, the air between the two surfaces expands and the layer becomes thicker. Note that the mass between the two surfaces remains the same.

In weather forecasting offices, charts showing isolines of constant thickness of the layer from 1000 *mb* to 500 *mb* are used to depict the regions of cold and warm air in the lower troposphere (see Figure 5.3).

Thickness charts can be readily constructed using data from radiosonde soundings of pressure, temperature and humidity at a network of upper air station.

Reduction of pressure to sea level

The variation of pressure with height is much greater than the horizontal variation across weather systems. This is so much so, that over hilly of mountainous terrain, a map of surface pressure would look like a contour map of the terrain variation. Therefore to isolate the pressure fluctuations associated with the passage of weather systems it is necessary to reduce all pressures to a common reference level, usually *mean sea level*.

One method would be to assume that an isothermal layer of air exists below the station level. Then if the (geopotential) height of the station is z_{st}, the station pressure p_{st} and the station temperature $\overline{T}_{st}$, the mean sea level pressure $p(0)$ may be estimated by integrating Eq. (5.1); i.e.,

$$p(0) = p_{st} \exp(z_{st} / H), \tag{5.9}$$

where $H = R_d \overline{T}_{st} / g$. If z_{st} is less than a few hundred metres, i.e., $z_{st} / H << 1$, a tolerable approximation to Eq. (5.9) is

$$p(0) = p_{st}[1 + (g\, z_{st} / R_d \overline{T}_{st})].$$

(Recall that $\exp(x) = 1 + x + ...$ for $x << 1$.) For larger station heights, the assumption of a mean isothermal layer below the station becomes questionable and empirical corrections are applied. Over mountainous terrain, the reduction to sea level pressure is not entirely satisfactory.

Figure 5.3 shows the mean sea level pressure (MSLP) and the 1000-500 mb thickness for a typical summertime situation in Australia. The main features include the surface cold front over southeastern Australia sandwiched between two anticyclones. The combination of MSLP and 1000-500mb thickness patterns show hot continental northerlies precede the front and cooler maritime southwesterlies follow it.

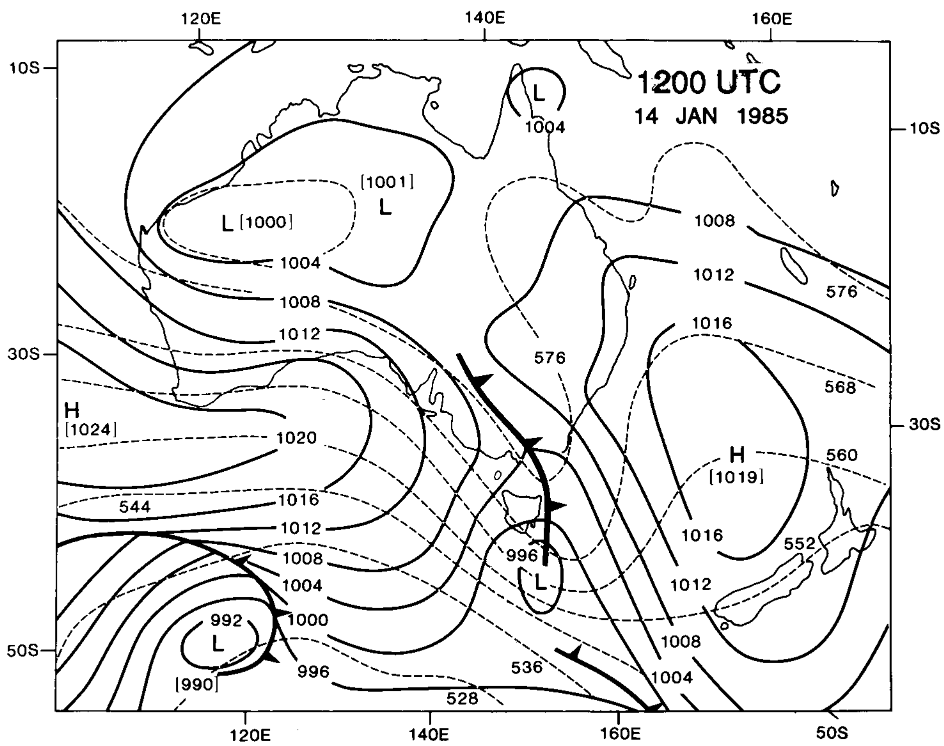

Figure 5.3. Mean sea level pressure (MSLP) (solid lines) and the 1000-500 mb thickness (dashed lines). The labels on the thickness contours are in decametres (10 m intervals), and hence the contour interval is 80 m.