7. Equilibrium and stability

Assume that the temperature varies with height, and suppose that a parcel with volume V is lifted a small distance *dz* from its equilibrium position at point *A* to a neighbouring point B (see Fig. 7.1). For $dz > 0$, the gas pressure at $z+dz$ is less than that at height z. Thus, as the particle rises, it expands and cools. In contrast, if it is displaced downwards, it is compressed and heated.

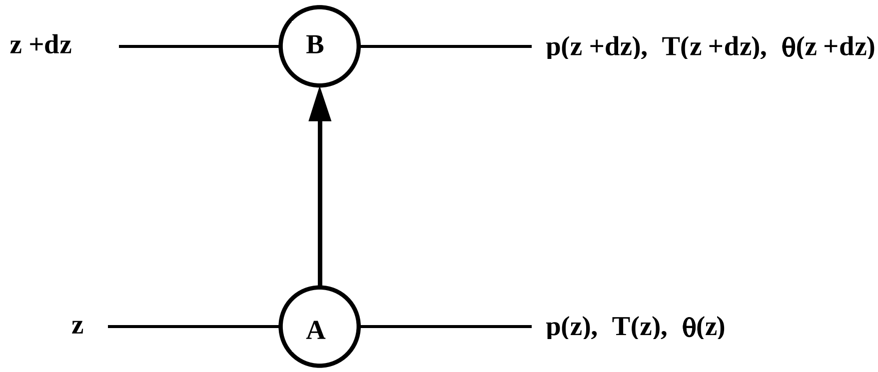

Figure 7.1. An air parcel lifted from is equilibrium position.

Although the displaced parcel's temperature and pressure change, its potential temperature remains constant and equal to its original value at *A*. Since the pressure at level *B* is $p(z + dz)$, the temperature of the parcel after displacement from A to *B*, given by the conservation of potential temperature,

$$T_B = \theta(z)\left[\, p(z+dz)/\, p_* \right]^{\kappa} . \tag{7.1}$$

Again using the definition of potential temperature, the temperature of the parcel's environment at level *B* is

$$T(z+dz) = \theta(z+dz)\left[\, p(z+dz)/\, p_* \right]^{\kappa} . \tag{7.2}$$

In general, the displaced parcel will experience a nonzero force at its new position. According to Archimedes principle, the force per unit mass, *F*, experienced by the parcel at *B* is

$$F \quad = \quad \frac{(\text{weight of air displaced - weight of air in parcel})}{\text{mass of air in parcel}}$$

$$= \frac{\left(g\rho(z+dz)V - g\rho_B V \right)}{\rho_B V} ,$$

where ρ_B is the parcel's density at level *B*. F, is called the *buoyancy force per unit mass*. Cancelling *V* and using the perfect gas law $\rho = p(z+dz)/RT$, the above expression gives

$$F = g\left(T_B - T(z+dz)\right) / T(z+dz) .$$

Using Eqs. (7.1) and (7.2), this becomes

$$F = g\left(\theta(z) - \theta(z+dz)\right) / \theta(z+dz) .$$

Now *F* is expressed entirely in terms of θ which is a measurable property of the environment, and can be written approximately as

$$F \approx -\frac{g}{\theta}\frac{d\theta}{dz}dz = -N^2 dz . \qquad (7.3)$$

If the buoyancy force (per unit mass) is directed from B to A (i.e., in the opposite direction to the parcel's displacement), it will push the parcel back towards point A. In this case the atmosphere is *stable* to small parcel displacements. Conversely, if *F* is directed from A to B, the parcel will accelerate away from point A, and therefore the atmosphere is *unstable* to small parcel displacements. Thus, the criterion for stability in a compressible atmosphere depends on the sign of the vertical gradient of potential temperature.

Returning to the displaced parcel and the case of stable stratification, assume that the subsequent particle motion is caused only by the buoyancy force and that the effect of the surrounding fluid can be neglected. We may then write down Newton's second Law for the parcel, i.e.,

acceleration = force per unit mass,

i.e.,

$$\frac{d^2\xi}{dt^2} + N^2\xi = 0, \qquad (7.4)$$

where

$$N^2 = \frac{g}{\theta}\frac{d\theta}{dz}, \qquad (7.5)$$

and $\xi(t)$ now denotes the displacement *dz.*

If the potential temperature increases with increasing *z,* then N^2 is a positive constant and Eq. (7.4) is the simple harmonic equation. In this case, it has a solution of form

$\xi = \xi_0 \cos(Nt + \chi)$, where ξ_0 and χ are constants. Thus, the air parcel oscillates with frequency N called the Brunt-Väisälä frequency or simply the buoyancy frequency. The corresponding period, $2\pi / N$, is called the Brunt-Väisälä period. On the other hand, if the potential temperature decreases with height, N^2 is negative and Eq. (7.4) has an exponentially-growing solution proportional to *exp* (Nt). Thus, when $N^2 > 0$, the atmosphere is stable to small displacements, whereas the atmosphere is unstable to small displacements when $N^2 < 0$

In general, N is a function of height. Nonetheless, the character of the solution to Eq. (7.4) is still oscillatory if $N^2 > 0$ and has exponential behaviour if $N^2 < 0$.

It is not surprising that N turns out to be a key parameter in the theory of gravity waves in the atmosphere, for it characterizes degree of gravitational stability

In general, for our stratified fluid calculations we assumed that the parcel motion was adiabatic, i.e., that heat diffusion into the parcel from its surrounding is "slow". We neglected heating or cooling on the time scale of a parcel to oscillate one period, i.e., $2\pi / N$. In the troposphere, this period is typically 10 minutes.

Except possibly close to the ground on a sunny day, unstable layers are not observed in the atmosphere. Even a slight degree of instability becomes neutrally stable, i.e., $d\theta \ / dz = 0.$ We refer to this mixed-up state as one of neutral stability.

During the day, when the ground is heated by solar radiation, the air layers adjacent to the ground are constantly being overturned by thermal convection to give a neutrally-stable well-mixed layer with a uniform potential temperature. In dry desert regions in the subtropics, the convective layer may be up to 4 km deep. In Melbourne on a warm summers day it is on the order of 1-2 km, depending on the surface moisture. If the ground surface is very moist, much of the sun's heat goes into evaporating water vapour, reducing the amount available to heat the air layer near the ground. At night, if the wind is not too strong, and especially if the air is dry and there is a clear sky, a strong *radiation inversion* forms in the lowest layers. An *inversion* is a layer of air in which not only the potential temperature, but also the temperature increases with height; such a layer is very stable. Typically, nocturnal radiation inversions are only a few tens of metres deep.

Vertical profiles of potential temperature

Figure 7.2 shows vertical profiles of potential temperature for a sequence of radiosonde soundings. The soundings span the passage of a cold front though Mt Isa. The front arrived at Mt Isa at 0940 EST 10 September 1991. The soundings at 2100 EST 9 September and at 0600 and 0900 EST 10 September are therefore about 13, 3 and 1 hours before the arrival of the front. The soundings at 1200 and 1500 EST are about 2 and 5 hours after the passage of the front respectively.

Only the lowest 5 km is shown. The unstable layer (negative $d\theta / dz$) near the ground during the day caused by strong insolation. The layer of almost constant θ above this layer is evidence that convective mixing is taking place. Above the neutral layer the profile is stable. By evening (2100) the atmosphere has cooled markedly in the lowest 100-200 m. The sounding at 1500, almost five hours after the passage of the cold front, there has been a very marked reduction in temperatures below 1.5 km.

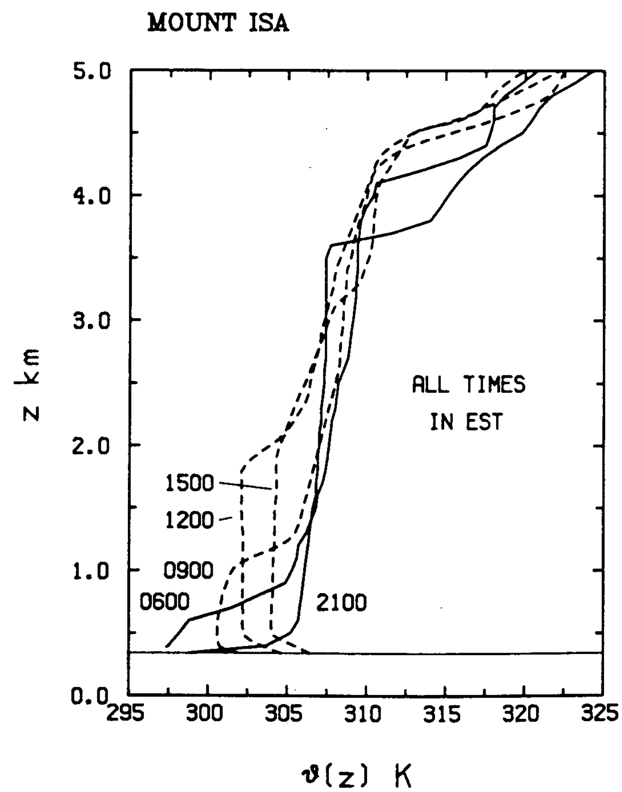

Figure 7.2. Vertical profile of θ for five radiosonde soundings made at Mt Isa.

Lapse rates

We saw earlier that on average, the temperature in the troposphere decreases with height. The magnitude of the rate of decrease is called the *temperature lapse rate*, or just the *lapse rate*. It is sometimes denoted by $\Gamma(= -dT / dz)$.

Consider now the lapse rate in an atmosphere in which the potential temperature is constant. From the definition of θ in Eq. (6.12), we have

$$\ln T = \ln \theta + \kappa (\ln p - \ln p_*).$$

Differentiating with respect to z gives

$$\frac{1}{T}\frac{dT}{dz}=\frac{1}{\theta}\frac{d\theta}{dz}+\frac{\kappa}{p}\frac{dp}{dz}.$$

Using the hydrostatic relation ($dp/dz=-g\rho$), the perfect gas law, and remembering that $\kappa = R/c_p$, gives

$$\frac{dT}{dz}=\frac{T}{\theta}\frac{d\theta}{dz}-\frac{g}{c_p}.$$

However, if θ is constant with height and if the air layer in question is in hydrostatic equilibrium, $d\theta/dz=0$, and hence

$$\frac{dT}{dz}=-\frac{g}{c_p}\ , \qquad (7.6)$$

This gives the rate at which temperature falls with height in a layer of dry air that has uniform potential temperature; for example the convectively well-mixed layer discussed above. It is very nearly 10°C per kilometre.

The same calculation applies to the change of temperature of an air parcel rising dry adiabatically. Accordingly, the temperature of such a parcel decreases with height at the dry adiabatic lapse rate.

Concept of an air parcel

From time to time in our discussion so far we have talked about an "air parcel" and it is appropriate that we clarify this concept. In many fluid mechanics problems, mixing is important only within a centimetre of the earth's surface and at levels above the turbopause (~105 km). At intermediate levels virtually all the vertical mixing is accomplished by the exchange of well-defined air parcels with horizontal dimensions ranging from a few centimetres to the scale of the earth itself.

We have sought to gain insights into the nature of vertical mixing in the atmosphere by considering the behaviour of an air parcel of infinitesimal dimensions that is assumed to be:

- thermally insulated from its environment so that its temperature changes adiabatically as it rises or sinks,
- always at exactly the same pressure as the environmental air at the same level, which is assumed to be in hydrostatic equilibrium, and
- moving slowly enough that its kinetic energy is a negligible fraction of its total energy.

In the case of real air parcels one or more of these assumptions is nearly always violated to some extent. However, this simple idealized model is helpful in understanding some of the physical processes that influence the distribution of vertical motion and vertical mixing in the atmosphere.