Lecture 6. Monin-Obuhkov similarity theory (Garratt 3.3)

Because so many BL measurements are made within the surface layer (i. e. where wind veering with height is insignificant) but stratification effects can be important at standard measurement heights of 2 m (for temperature and moisture) and 10 m (for winds), it is desirable to correct the log-layer profiles for stratification effects.

Based on the scaling arguments of last lecture, Monin and Obuhkov (1954) suggested that the vertical variation of mean flow and turbulence characteristics in the surface layer should depend only on the surface momentum flux as measured by friction velocity u_*, the buoyancy flux B_0, and the height z. One can form a single nondimensional combination of these, which is traditionally chosen as the **stability parameter**

$$\zeta = z/L$$

The logarithmic scaling regime of last time corresponds to $\zeta << 1$.

Thus, within the surface layer, we must have

$$(kz/u_*)(\partial\bar{u}/\partial z) = \phi_m(\zeta) \tag{1}$$

$$(kz/\bar{\theta}_*)(\partial\bar{\theta}/\partial z) = \phi_h(\zeta) \tag{2}$$

where $\phi_m(\zeta)$ and $\phi_h(\zeta)$ are **universal similarity functions** which relate the fluxes of momentum and θ (i. e. sensible heat) to their mean gradients. Other adiabatically conserved scalars should behave similarly to θ since the transport is associated with eddies which are too large to be affected by molecular diffusion or viscosity. To agree with the log layer scaling, $\phi_m(\zeta)$ and $\phi_h(\zeta)$ should approach 1 for small ζ.

We can express (1) and (2) in other equivalent forms. First, we can regard them as defining surface layer **eddy viscosities**:

$$K_m = -\overline{u'w'} \,/\, (\partial\bar{u}/\partial z) = u_*^2/(\phi_m(\zeta)\, u_*/kz) = ku_*z \,/\, \phi_m(\zeta)$$

$$K_h = -\overline{w'\theta'} \,/\, (\partial\bar{\theta}/\partial z) = u_*\theta_*/(\phi_h(\zeta)\, \theta_*/kz) = ku_*z \,/\, \phi_h(\zeta)$$

By analogy to the molecular Prandtl number, the **turbulent Prandtl number** is their ratio:

$$\mathrm{Pr}_t = K_m / K_h = \phi_h(\zeta) \,/\, \phi_m(\zeta)$$

Another commonly used form of the similarity functions is to measure stability with gradient Richardson number Ri instead of ζ. Recalling that $N^2 = -d\bar{b}/dz$, and again noting that the surface layer is thin, so vertical fluxes do not vary significantly with height within it, Ri is related to ζ as follows:

$$\begin{aligned}\mathrm{Ri} &= (-d\bar{b}/dz) \,/\, (d\bar{u}/dz)^2 \\ &= (\overline{w'b'}_0 /K_h)/ (\,\overline{u'w'}_0/K_m)^2 \\ &= (B_0\phi_h(\zeta) /ku_*z) \,/\, (u_*^2\phi_m(\zeta)/ku_*z))^2 \\ &= \zeta\phi_h/\phi_m^2\end{aligned}$$

Given expressions for $\phi_m(\zeta)$ and $\phi_h(\zeta)$, we can write ζ and hence the similarity functions and eddy diffusivities in terms of Ri. The corresponding formulas for dependence of eddy diffusivity on Ri (stability) are often used by modellers even outside the surface layer, with the neutral K_m and K_m

estimated as the product of an appropriate velocity scale and lengthscale.

Field Experiments

The universal functions must be determined empirically. In the 1950-60s, several field experiments were conducted for this purpose over regions of flat, homogeneous ground with low, homogeneous roughness elements, culminating in the 1968 Kansas experiment. This used a 32 m instrumented tower in the middle of a 1 mi^2 field of wheat stubble. Businger et al. (1971, JAS, **28**, 181-189) documented the relations below, which are still accepted:

$$\phi_m = \begin{cases} (1-\gamma_1\zeta)^{-1/4}, & \text{for } -2 < \zeta < 0 \text{ (unstable)} \\ 1+\beta\zeta, & \text{for } 0 \le \zeta < 1 \text{ (stable)} \end{cases}$$

$$\phi_h = \begin{cases} \mathrm{Pr}_{tN}(1-\gamma_2\zeta)^{-1/2}, & \text{for } -2 < \zeta < 0 \text{ (unstable)} \\ \mathrm{Pr}_{tN}+\beta\zeta, & \text{for } 0 \le \zeta < 1 \text{ (stable)} \end{cases}$$

The values of the constants determined by the Kansas experiment were

$$\mathrm{Pr}_{tN} = 0.74,\ \beta = 4.7,\ \gamma_1 = 15,\ \gamma_2 = 9\ .$$

The quality of the fits to observations are shown on the next page. Other experiments have yielded somewhat different values of the constants (Garratt, Appendix 4, Table A5), so we will follow Garratt (p. 52) and Dyer (1974, *Bound-Layer Meteor*., **7**, 363-372*)* and assume:

$$\mathrm{Pr}_{tN} \approx 1,\ \beta \approx 5,\ \gamma_1 \approx \gamma_2 \approx 16$$

In neutral or stable stratification, this implies $\phi_m = \phi_h$, i. e. pressure perturbations do not affect the eddy transport of momentum relative to scalars such as heat, and the turbulent Prandtl number is 1. In unstable stratification, the eddy diffusivity for scalars is more than for momentum.

Solving these relations for Ri,

$$\zeta = \begin{cases} \mathrm{Ri}, & \text{for } -2 < \mathrm{Ri} < 0 \text{ (unstable)} \\ \dfrac{\mathrm{Ri}}{1-5\mathrm{Ri}}, & \text{for } 0 \le \mathrm{Ri} < 0.2 \text{ (stable)} \end{cases}$$

Limiting cases (Garratt, p. 50)

(i) Neutral limit. ϕ_m , $\phi_h \to 1$ as $\zeta \to 0$ as expected, recovering log-layer scaling for $z << |L|$.

(ii) Stable limit. Expect eddy size to depend on L rather than z (z -less scaling), since our scaling analysis of last time suggests that stable buoyancy forces tend to suppress eddies with a scale larger than L. This implies that the eddy diffusivity

$$K_m = ku_*z/\phi_m \propto \text{(velocity)(length)} \propto u_*L \quad \Rightarrow \quad \phi_m \sim z/L = \zeta$$

and similarly for K_h. The empirical formulas imply $K_m \sim \beta\zeta$ for large ζ, which is consistent with this limit. Hence they are usually assumed to apply for all positive ζ.

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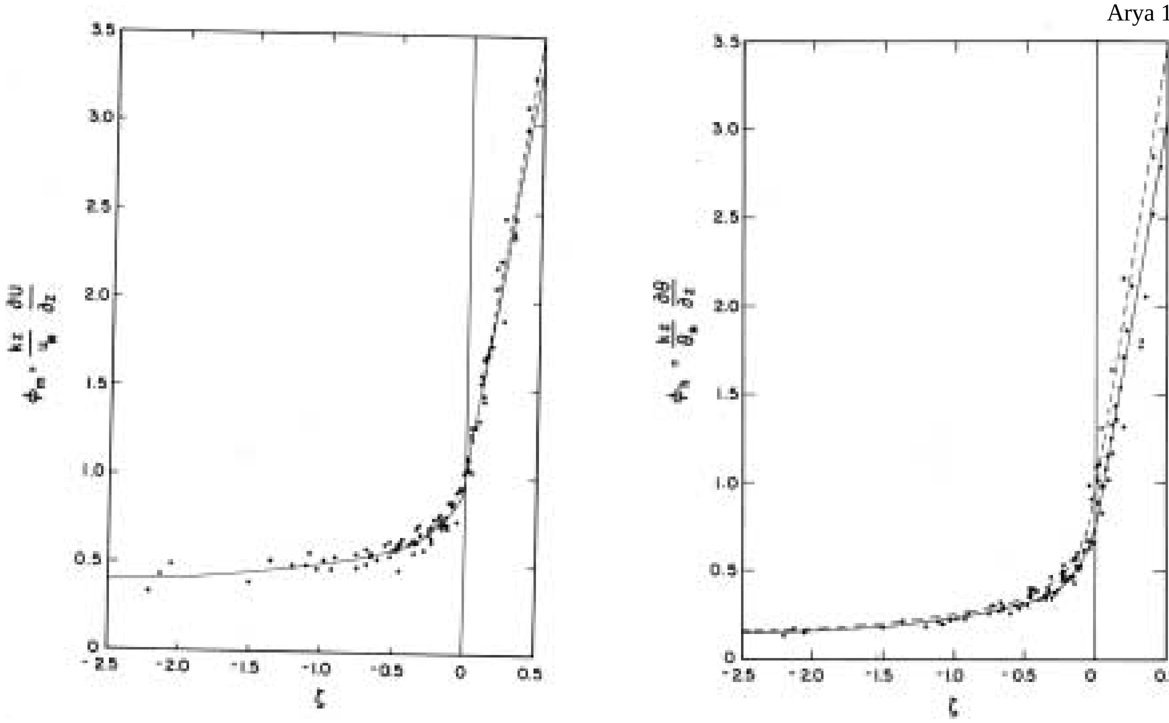

Empirical determination of similarity functions from Kansas experiment

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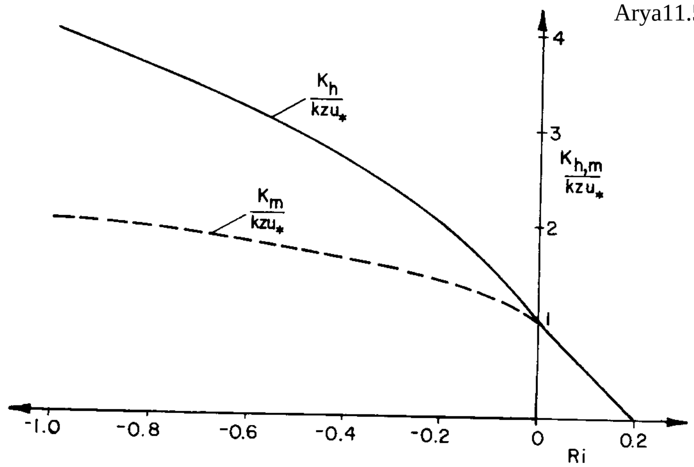

Eddy viscosity and diffusivity as functions of stability, measured by Ri

(iii) Unstable limit. Convection replaces shear as the main source of eddy energy, so we expect the eddy velocity to scale with the buoyancy flux B_0 and not the friction velocity. We still assume that the eddy size is limited by the distance z to the boundary. In this ‘free convective scaling’, the eddy velocity scale is $u_f = (B_0 z)^{1/3}$ and the eddy viscosity should go as

$$K_m = ku_* z/\phi_m \propto u_f z \quad \Rightarrow \quad \phi_m \propto u_*/u_f \propto (-z/L)^{-1/3} = (-\zeta)^{-1/3}$$

A similar argument applies to eddy diffusivity for scalars K_h. The empirical relations go as $(-\zeta)^{-1/2}$ for scalars and $(-\zeta)^{-1/4}$ for momenta, but reliable measurements only extend out to $\zeta = -2$. Free convective scaling may be physically realized, but only at higher ζ.

Wind and thermodynamic profiles

The similarity relations can be integrated with respect to height to get:

$$\bar{u}/u_* = k^{-1} [\ln(z/z_0) - \psi_m(z/L)]$$

$$(\theta_0 - \bar{\theta})/\theta_* = k^{-1} [\ln(z/z_{T0}) - \psi_h(z/L)] \text{ (and similarly for other scalars)}$$

where if $x = (1 - \gamma_1\zeta)^{1/4}$,

$$\psi_m(\zeta) = \int_0^{\zeta} [1 - \phi_m(\zeta')] d\zeta'/\zeta'$$

$$= \begin{cases} \ln\left(\left(\dfrac{1+x^2}{2}\right)\left(\dfrac{1+x}{2}\right)^2\right) - 2\tan^{-1}x + \dfrac{\pi}{2}, & \text{for } -2 < \zeta < 0 \text{ (unstable)} \\ -\beta\zeta, & \text{for } 0 \le \zeta \text{ (stable)} \end{cases}$$

$$\psi_m(\zeta) = \int_0^{\zeta} [1 - \phi_h(\zeta')] d\zeta'/\zeta'$$

$$= \begin{cases} 2\ln\left(\dfrac{1+x^2}{2}\right), & \text{for } -2 < \zeta < 0 \text{ (unstable)} \\ -\beta\zeta, & \text{for } 0 \le \zeta \text{ (stable)} \end{cases}$$

Wind profiles in stable, neutral, and unstable conditions are shown in the figure below. Low-level wind and shear are reduced compared to the log profile in unstable conditions, when K_m is larger. From these,we derive bulk aerodynamic coefficients which apply in non-neutral conditions:

$$C_D = \frac{k^2}{[\ln(z/z_0) - \psi_m(z/L)]^2}, \quad C_H = \frac{k^2}{[\ln(z/z_{T0}) - \psi_m(z/L)][\ln(z/z_{T0}) - \psi_h(z/L)]} \quad (3)$$

These decrease considerably in stable conditions (see figure on next page). In observational analyses and numerical models, (3) and the formula for L are solved simultaneously to find surface heat and momentum fluxes from the values of u and $\theta_0 - \theta$ at the measurement or lowest grid-level z

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Garratt

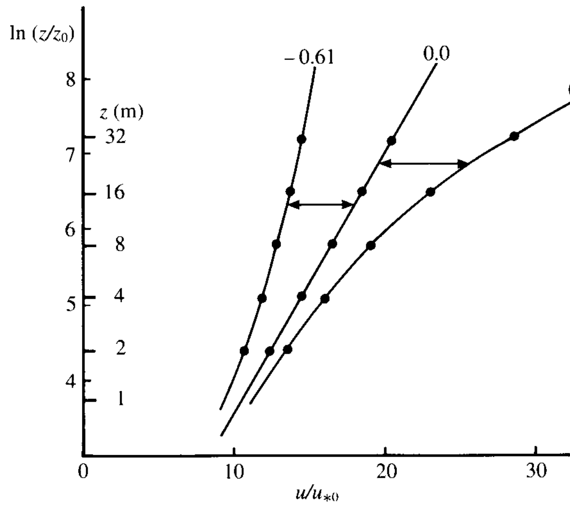


Fig. 3.5 Three wind profiles from the Kansas field data (Izumi, 1971) plotted in normalized form at three values of the gradient Ri ($z = 5.66$ m). Both normalized and absolute heights are shown, whilst the magnitude of the horizontal arrows indicates the effect of buoyancy on the wind relative to the neutral profile (see Eq. 3.34).

Garratt

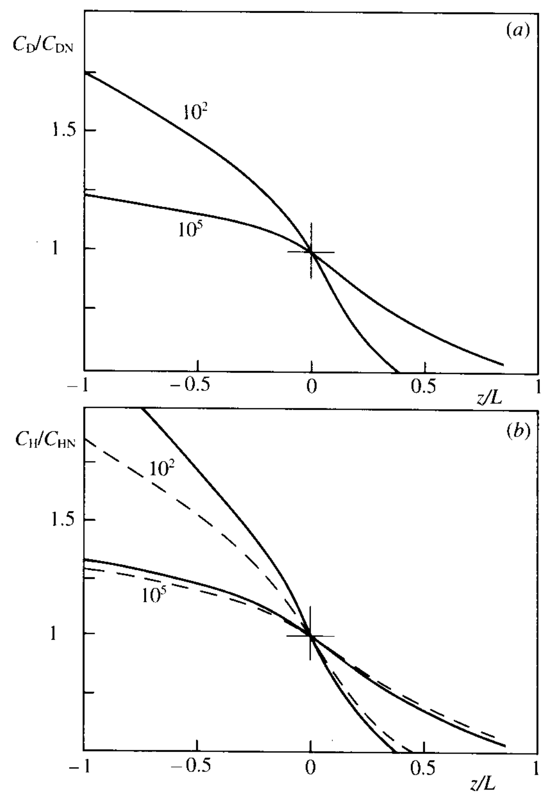


Fig. 3.7 Values of (a) C_D/C_{DN} and (b) C_H/C_{HN} as functions of z/L for two values of z/z_0 as indicated. In (b), the solid curves have $z_0 = z_T$, and the pecked curves have $z_0/z_T = 7.4$ (see Chapter 4).

Scaling for the entire boundary layer (Garratt, 3.2)

In general, the BL depth h and turbulence profile depend on many factors, including history, stability, baroclinicity, clouds, presence of a capping inversion, etc. Hence universal formulas for the velocity and thermodynamic profiles above the surface layer (i. e. where transports are primarily by the large, BL-filling eddies) are rarely applicable.

However, a couple of special cases are illuminating to consider. The first is a **well-mixed** BL (homework), in which the fluxes adjust to ensure that the tendency of θ, q, and velocity remain the same at all levels. Well mixed BLs are usually either strongly convective, or strongly driven stable BLs capped by a strong inversion. Mixed layer models incorporating an **entrainment closure** for determining the rate at which BL turbulence incorporates above-BL air into the mixed layer are widely used.

The other interesting (though rarely observable) case is a steady-state, neutral, barotropic BL. This is the turbulent analogue to a laminar Ekman layer. Here, the fundamental scaling parameters are $G = |\mathbf{u}_g|$, f, and z_0. Out of these one can form one independent nondimensional parameter, the surface Rossby number $\mathrm{Ro}_s = G/fz_0$ (which is typically 10^4 -10^8). The friction velocity (which measures surface stress) must have the form

$$u_* / G = F(\mathrm{Ro}_s)$$

Hence, one can also regard u_* / G as a proxy nondimensional control parameter in place of Ro_s. The steady-state BL momentum equations are

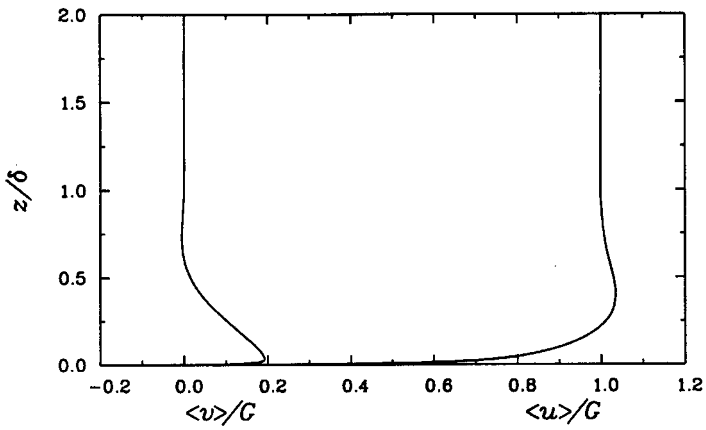
$$f(u - u_g) = -\frac{\partial}{\partial z}\overline{v'w'}$$

$$f(v - v_g) = \frac{\partial}{\partial z}\overline{u'w'}$$

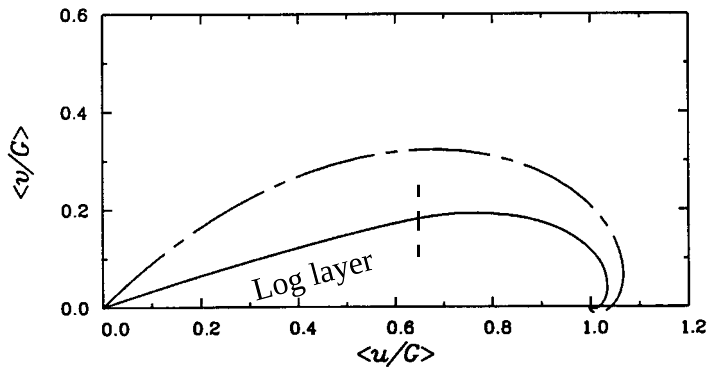
On the next page are velocity and momentum flux profiles from a direct numerical simulation (384×384×85 gridpoints) in which $u_*/G = 0.053$ (Coleman 1999, *J. Atmos. Sci*, **56**, 891-900). The geostrophic wind is oriented in the x direction, and is independent of height (the barotropic assumption). Height is nondimensionalized by $\delta = u_*/f$. In the thin surface layer, extending up to $z = 0.02\delta$, the wind increases logarithmically with height without appreciable turning (this is most clearly seen on the wind hodograph), and is turned at 20° from geostrophic (this angle is an increasing function of u_*/G) The neutral BL depth , defined as the top of the region of significantly ageostrophic mean wind, is

$$h_N = 0.8u_*/f$$

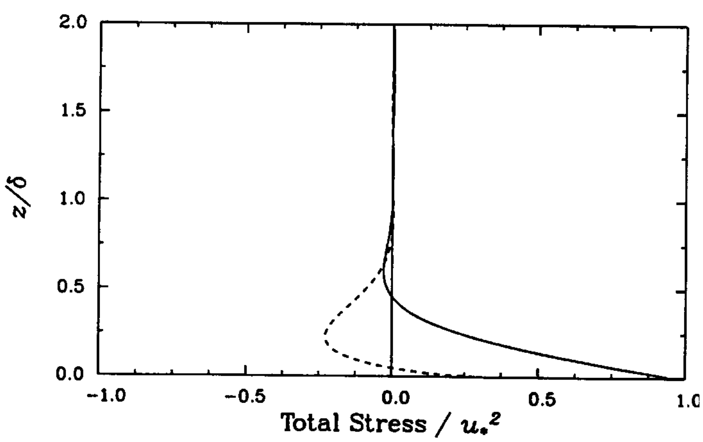
For $u_* = 0.3$ m s^{-1} and $f = 10^{-4}$ s^{-1}, $h_N = 2.4$ km. Real ABLs are rarely this deep because of stratification aloft, but fair approximations to the idealized turbulent Ekman layer can occur in strong-winds over the midlatitude oceans. The wind profile qualitatively resembles an Ekman layer of with an Ekman thickness $(2\nu/f)^{1/2} = 0.12u_*/f$, except much more of the wind shear is compressed into the surface layer.



Wind profiles in a neutral barotropic BL with
u_*/G = 0.053 (Coleman 1999).



Wind hodograph (dashed = Ekman layer). Log (surface) layer is part of profile to right of dashes.



Stress profiles in geostrophic coordinate system.
Solid = in direction of $\mathbf{u}_g$, dashed = transverse dir.

The wind and momentum flux profiles depend weakly on Ro_S, but we will describe below a scaling that collapses these into a single universal profile independent of Ro_S above the surface layer As we go up through the boundary layer, the magnitude of the momentum flux will decrease from u_*^2 at the surface to near zero at the BL top, so throughout the BL, the momentum flux will be $O(u_*^2)$(Hence, throughout the BL the turbulent velocity perturbations u', w' should scale with u_* to be consistent with this momentum flux). We assume that the BL depth scales with u_*/f. These scalings suggest a nondimensionalization of the steady state BL momentum equations:

$$\frac{u-u_g}{u_*} = -\frac{\partial(\overline{v'w'}/u_*^2)}{\partial(zf/u_*)}$$

$$\frac{v-v_g}{u_*} = \frac{\partial(\overline{u'w'}/u_*^2)}{\partial(zf/u_*)}$$

If we adopt a coordinate system in which the x axis is in the direction of the surface-layer wind, the boundary conditions on the momentum flux are

$$\overline{u'w'}/u_*^2 = -1, \overline{v'w'}/u_*^2 = 0 \quad \text{as } z \to 0 \text{ (at surface layer top)}$$

$$\overline{u'w'}/u_*^2 = 0\ , \ \overline{v'w'}/u_*^2 = 0 \quad \text{as } z \to \infty$$

This momentum balance and boundary conditions are consistent with a universal **velocity defect law** of the form:

$$(u-u_g)/u_* = F_x(zf/u_*)$$

$$(v-v_g)/u_* = F_y(zf/u_*)$$

where F_x and F_y are universal functions (which must be determined empirically or via LES simulation) that apply for any Ro_S. In the surface layer , these universal functions cease to apply and the logarithmic wind profile $u/ku_* = \ln(z/z_0)$, $v = 0$ matches onto the defect laws. The figure below shows that Coleman’s simulations and laboratory experiments with different parameters are consistent with the same F_x and F_y, supporting their universality.