METR 4433 – Mesoscale Meteorology
Spring 2001

Problem Set #1 Answers
Scales and PBL

Distributed Friday February 9, 2001
Due 5pm, Wednesday February 21, 2001

1. Using the three momentum equations, perform a scale analysis for a supercell thunderstorm and describe the main characteristics of motion at this scale.

For a supercell storm,

$U \sim W \sim 10$ m/s,
$L \sim H \sim 10$ km $= 10^4$ m
$T_H \sim L/U = 10^3$ s
$T_V \sim H/W = 10^3$ s
$\Delta P \sim 1$ mb = 100Pascal
$\rho \sim 1$ kg/m^3
$f \sim 10^{-4}$ s
$\theta_0 \sim 300$ K
$\Delta\theta \sim 1$ K

We use Boussinesq approximated equations for scale analysis.

For the horizontal equation, we look at the x component of equation:

$$\begin{array}{ccccc}
\dfrac{\partial u}{\partial t} \quad + & u\dfrac{\partial u}{\partial x} + & w\dfrac{\partial u}{\partial z} = & -\dfrac{1}{\bar{\rho}}\dfrac{\partial p'}{\partial x} & +fv \\
\dfrac{V}{T_H} & \dfrac{VV}{L} & \dfrac{WV}{H} & \dfrac{\Delta p}{\rho L} & fV \\
\dfrac{10}{10^3} & \dfrac{10^2}{10^4} & \dfrac{10^2}{10^4} & \dfrac{10^2}{10^4} & 10^{-4}(10) \\
10^{-2} & 10^{-2} & 10^{-2} & 10^{-2} & 10^{-3}
\end{array}$$

Therefore the Coriolis term is one order of magnitude smaller than the other terms, it can therefore be neglected. The other terms are of the same order of magnitude therefore the local time tendency, advection and PGF are all of equal importance

For the vertical equation:

$$
\begin{array}{ccccccccc}
\dfrac{\partial w}{\partial t} & + & u\dfrac{\partial w}{\partial x} + & w\dfrac{\partial w}{\partial z} = & -\dfrac{1}{\bar{\rho}}\dfrac{\partial p'}{\partial z} & + & g\dfrac{\theta'}{\bar{\theta}} \\
\dfrac{V}{T_V} & & \dfrac{VV}{L} & \dfrac{WV}{H} & \dfrac{\Delta P}{\rho H} & & \dfrac{g\Delta\theta}{\theta_0} \\
\dfrac{10}{10^3} & & \dfrac{10^2}{10^4} & \dfrac{10^2}{10^4} & \dfrac{10^2}{10^4} & & \dfrac{10}{300} \\
10^{-2} & & 10^{-2} & 10^{-2} & 10^{-2} & & 3.3\times10^{-2}
\end{array}
$$

Therefore all terms are of equal order of magnitude, none of them can be neglected. The buoyancy terms appears to be a dominant terms, not surprisingly for convective system.

The motion is ageostrophic, nonhydrostatic, and three dimensional. The effect of earth rotation can be neglected (sometimes called irrotational).

2. Given the following instantaneous measurement of potential temperature (θ) and vertical velocity (w) in this table, fill in all the remaining blanks in the table. Also verify with the answers from the above that $\overline{w\theta} = \bar{w}\bar{\theta} + \overline{w'\theta'}$.

Measurement			Calculations					
Index	w	θ	w'	θ'	$(w')^2$	$(\theta')^2$	wθ	w'θ'
1	0.5	295	0.38	1	0.1444	1	147.5	0.38
2	-0.5	293	-0.62	-1	0.3844	1	-146.5	0.62
3	1.0	295	0.88	1	0.7744	1	295	0.88
4	0.8	298	0.68	4	0.4624	16	238.4	2.72
5	0.9	292	0.78	-2	0.6084	4	262.8	-1.54
6	-0.2	294	-0.32	0	0.1024	0	-0.58	0
7	-0.4	292	-0.52	-2	0.2704	4	-116.8	1.04
8	0.1	289	-0.2	-5	0.0004	25	28.9	0.1
9	-0.9	293	-1.02	-1	1.0404	1	-263.7	1.02
10	-0.1	299	-0.22	5	0.0484	25	-29.9	-1.1
Average	0.12	294	0	0	0.3836	7.8	35.69	0.41

$\bar{w}\bar{\theta} + \overline{w'\theta'} = 0.12\times294+0.41=35.28+0.41=35.69 = \overline{w\theta}$.

$\overline{w\theta} = \bar{w}\bar{\theta} + \overline{w'\theta'}$ is verified by our calculation!

Based on the turbulent flux your obtain, is the data characteristic of a stable, neutral or unstable boundary layer? (Hint, think of the net heat transport in an unstable boundary layer – should the net transport be upward or downward?).

Unstable, because the flux is positive so w' and θ' are positively correlated. It means that the on average, warmer air is transported upward by the turbulent eddies and this happens in unstable environment.

3. Problem 5.5 on page 139 of Holton.

Suppose that in a certain region the geostrophic wind is westerly at 15 m/s. Compute the net cross-isobaric transport in the planetary boundary layer using both the mixed-layer solution (5.22) and the Ekman layer solution (5.31). You may let $|\overline{\vec{V}}|$ = u_g in (5.22), h = De = 1 km, k_s = 0.015 m^{-1} s and ρ = 1 kg m^{-3}.

The cross-isobaric flux is defined as

$$M = \int_0^h \rho_0 \overline{v} dz \qquad (1)$$

therefore we need to know $\overline{v}$.

The mixed layer solution is:

$$\overline{u} = \overline{u}_g - K_s |\overline{\vec{V}}| \overline{v}$$
$$\overline{v} = K_s |\overline{\vec{V}}| \overline{u}$$

From the solution, we can solve for $\overline{v}$:

$$\overline{v} = K_s |\overline{\vec{V}}| (\overline{u}_g - K_s |\overline{\vec{V}}| \overline{v}) = K_s |\overline{\vec{V}}| \overline{u}_g - (K_s |\overline{\vec{V}}|)^2 \overline{v} \rightarrow$$
$$\overline{v}\left[1 + (K_s |\overline{\vec{V}}|)^2\right] = K_s |\overline{\vec{V}}| \overline{u}_g \rightarrow$$
$$\overline{v} = \frac{K_s |\overline{\vec{V}}| \overline{u}_g}{\left[1 + (K_s |\overline{\vec{V}}|)^2\right]}. \qquad (2)$$

We can see that $\overline{v}$ is constant with height, an assumption we made in the mixed-layer model. Substitute $\overline{v}$ in (2) into (1), we obtain the flux:

$$M = \int_0^h \rho_0 \frac{K_s |\overline{\vec{V}}| \overline{u}_g}{1 + (K_s |\overline{\vec{V}}|)^2} dz = \frac{\rho_0 K_s |\overline{\vec{V}}| \overline{u}_g h}{1 + (K_s |\overline{\vec{V}}|)^2} = \frac{1 \times 0.015 \times 15^2 \times 10^3}{1 + (0.015 \times 15)^2} \approx 3.21 \times 10^3 kg\, m^{-1} s^{-1}$$

For Ekman layer, the mass flux is defined as

$$M = \int_0^{D_e} \rho_0 \overline{u}_g \exp(-\gamma z) \sin(\gamma z) dz \quad \text{where } \gamma = \frac{\pi}{D_e}.$$

We need to integrate the above equation. Let $x = \gamma$z (you don't have to do this), M becomes:

$$M = \frac{\rho_0 \overline{u}_g}{\gamma} \int_0^{D_e\gamma} \exp(-x)\sin(x)dx\,.$$

Let $I = \int_0^{D_e\gamma} \exp(-x)\sin(x)dx$, integrate by parts →

$$\begin{aligned}
I &= \int_0^{D_e\gamma} \exp(-x)\sin(x)dx = -\int_0^{D_e\gamma} \exp(-x)d\cos(x) \\
&= -\left[\exp(-x)\cos(x)\right]_0^{D_e\gamma} + \int_0^{D_e\gamma} \cos(x)d\exp(-x) \\
&= -\left[\exp(-x)\cos(x)\right]_0^{D_e\gamma} - \int_0^{D_e\gamma} \exp(-x)\cos(x)dx \\
&= -\left[\exp(-x)\cos(x)\right]_0^{D_e\gamma} - \int_0^{D_e\gamma} \exp(-x)d\sin(x) \\
&= -\left[\exp(-x)\cos(x) + \exp(-x)\sin(x)\right]_0^{D_e\gamma} + \int_0^{D_e\gamma} \sin(x)d\exp(-x) \\
&= -\left[\exp(-x)\left\{\cos(x)+\sin(x)\right\}\right]_0^{D_e\gamma} - \int_0^{D_e\gamma} \exp(-x)\sin(x)dx \\
&= -\left[\exp(-x)\left\{\cos(x)+\sin(x)\right\}\right]_0^{D_e\gamma} - I
\end{aligned}$$

$$\begin{aligned}
I &= -\frac{1}{2}\left[\exp(-x)\left\{\cos(x)+\sin(x)\right\}\right]_0^{D_e\gamma} \\
&= -\frac{1}{2}\left[\exp(-x)\left\{\cos(x)+\sin(x)\right\}\right]_0^{\pi} \\
&= -\frac{1}{2}\left[\exp(-\pi)\left\{\cos(\pi)+\sin(\pi)\right\} - \exp(-0)\left\{\cos(0)+\sin(0)\right\}\right] \\
&= -\frac{1}{2}\left[-\exp(-\pi)-1\right] = \frac{1}{2}\left[\exp(-\pi)+1\right] \approx 0.5216
\end{aligned}$$

The final answer is

$$M = \frac{\rho_0 \overline{u}_g I}{\gamma} = \frac{1\times 15\times 0.5216\times 10^3}{3.14} \approx 2.49\times 10^3\, kg\, m^{-1} s^{-1}\,.$$

4. Problem 5.10 on page 140 of Holton.

Derive a formula for the vertical velocity at the top of the planetary boundary layer using the mixed-layer expression (5.22). Assume that |V| = 10 m/s is independent of x and y and the $\overline{u}_g = \overline{u}_g(y)$. If h = 1 km, what value of k_s have if the result is to agree with the vertical velocity derived from the Ekman layer solution with D_e = 1km?

From the mass continuity equation, we know that the w at the PBL top is

$$w(D_e) = -\int_0^{D_e} \left(\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} \right) dz \;. \tag{3}$$

With the mixed-layer model, u and v are constant with height, therefore

$$w(D_e) = -D_e \left(\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} \right). \tag{4}$$

In the previous problem, we obtained

$$\overline{v} = \frac{K_s \,|\overline{\vec{V}}|\, \overline{u}_g}{\left[1 + (K_s \,|\overline{\vec{V}}|)^2 \right]} \tag{5}$$

therefore

$$\begin{aligned} \overline{u} &= \overline{u}_g - K_s \,|\overline{\vec{V}}|\, \overline{v} = \overline{u}_g - \frac{(K_s \,|\overline{\vec{V}}|)^2 \overline{u}_g}{1 + (K_s \,|\overline{\vec{V}}|)^2} \\ &= \frac{\overline{u}_g + (K_s \,|\overline{\vec{V}}|)^2 \overline{u}_g - (K_s \,|\overline{\vec{V}}|)^2 \overline{u}_g}{1 + (K_s \,|\overline{\vec{V}}|)^2} = \frac{\overline{u}_g}{1 + (K_s \,|\overline{\vec{V}}|)^2} \end{aligned} \tag{6}$$

Substitute the $\overline{u}$ and $\overline{v}$ in (5) and (6) into (4), noting that $\overline{u}_g$ in independent of x, we get

$$w(D_e) = -D_e \frac{K_s \,|\overline{\vec{V}}|}{1 + (K_s \,|\overline{\vec{V}}|)^2} \frac{\partial \overline{u}_g}{\partial y} = D_e \frac{K_s \,|\overline{\vec{V}}|}{1 + (K_s \,|\overline{\vec{V}}|)^2} \xi_g \tag{7}$$

For the Ekman layer, according to Holton,

$$w(D_e) = \frac{1}{2\gamma} \xi_g = \frac{D_e}{2\pi} \xi_g \tag{8}$$

Equaling w in (7) and (8) gives

$$\frac{D_e}{2\pi} \xi_g = D_e \frac{K_s \,|\overline{\vec{V}}|}{1 + (K_s \,|\overline{\vec{V}}|)^2} \xi_g \rightarrow$$

$$1 + (K_s \,|\overline{\vec{V}}|)^2 = 2\pi K_s \,|\overline{\vec{V}}| \;\rightarrow$$

$$K_s^{\,2}\,|\overline{\overline{\vec{V}}}|^2 - 2\pi\,|\overline{\overline{\vec{V}}}|\,K_s + 1 = 0 \qquad (9)$$

(9) is a quadratic equation for k_s. Solve for $K_{s:}$

$$K_s = \frac{2\pi\,|\overline{\overline{\vec{V}}}| \pm \sqrt{(2\pi\,|\overline{\overline{\vec{V}}}|)^2 - 4\,|\overline{\overline{\vec{V}}}|^2}}{2\,|\overline{\overline{\vec{V}}}|^2} = \frac{\pi \pm \sqrt{\pi^2 - 1}}{|\overline{\overline{\vec{V}}}|} = 0.61 \; or \; 0.0163 \; m^{-1}s$$

Both values appear to be valid but the first value is probably more realistic.