Thunderstorm Dynamics

Houze sections 7.4, 8.3, 8.5, Refer back to equations in Section 2.3 when necessary.
Bluestein Vol. II section 3.4.6.
Review article "Dynamics of Tornadic Thunderstorms" by Klemp – handout.

A. Equations of Motion

Boussinesq approximated equations (neglecting friction and Coriolis force)

$$\frac{du}{dt} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x} \tag{1a}$$

$$\frac{dv}{dt} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial y} \tag{1b}$$

$$\frac{dw}{dt} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} + B \tag{1c}$$

where buoyancy B includes effects of air density and water loading. The prime is with respect to a horizontally homogeneous base state.

Written in a vector form:

$$\frac{d\vec{V}}{dt} = -\frac{1}{\rho_0}\nabla p' + B\hat{k} \tag{3}$$

or

$$\frac{\partial \vec{V}}{\partial t} = -\vec{V}\cdot\nabla\vec{V} - \frac{1}{\rho_0}\nabla p' + B\hat{k}. \tag{4}$$

Verify it for yourself that

$$\vec{V}\cdot\nabla\vec{V} = \nabla\left(\frac{\vec{V}\cdot\vec{V}}{2}\right) - \vec{V}\times\left(\nabla\times\vec{V}\right) \tag{4}$$

$$\vec{\omega} = \nabla\times\vec{V} = \text{3D vorticity}$$

therefore

$$\frac{\partial \vec{V}}{\partial t} = -\nabla\left(\frac{\vec{V}\cdot\vec{V}}{2} + \frac{p'}{\rho_0}\right) + \vec{V}\times\vec{\omega} + B\hat{k} \qquad (5)$$

B. Vorticity Equation

$$\vec{\omega} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}\ .$$

Derive 3-D voriticity equation by taking $\nabla\times(5)$ and recall $\nabla\times\nabla(\ ) = 0$ →

$$\frac{\partial \vec{\omega}}{\partial t} = \nabla\times(\vec{V}\times\vec{\omega}) + \nabla\times(B\hat{k}) \qquad (6)$$

1. Development of rotation (vertical vorticity)

To investigate the development of rotation in thunderstorms, look at the vertical component of vorticity $\varsigma = \hat{k}\cdot\vec{\omega}$ →

$$\frac{\partial \varsigma}{\partial t} = \hat{k}\cdot\nabla\times(\vec{V}\times\vec{\omega}) + \hat{k}\cdot\nabla\times(B\hat{k}) = \hat{k}\cdot\nabla\times(\vec{V}\times\vec{\omega})$$

Note that $\hat{k}\cdot\nabla\times(B\hat{k}) = 0$ (verify yourself) therefore buoyancy does not directly generate vertical rotation in thunderstorms! Buoyancy only generates horizontal vorticity which can be tilted into the vertical direction.

Let ξ, η and ς be the x, y and z component of vorticity, respectively.

$$\vec{V}\times\vec{\omega} = (v\varsigma - w\eta)\vec{i} + (w\xi - u\varsigma)\vec{j} + (u\eta - v\xi)\vec{k}$$

and

$$\hat{k}\cdot\nabla\times(\vec{V}\times\vec{\omega}) = \frac{\partial}{\partial x}(w\xi - u\varsigma) - \frac{\partial}{\partial y}(v\varsigma - w\eta)$$

$$\frac{\partial \varsigma}{\partial t} = -u\frac{\partial \varsigma}{\partial x} - v\frac{\partial \varsigma}{\partial y} - w\frac{\partial \varsigma}{\partial z} - \varsigma\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \xi\frac{\partial w}{\partial x} + \eta\frac{\partial w}{\partial y} \qquad (7)$$

or

$$\frac{\partial \varsigma}{\partial t} = -\vec{V} \cdot \nabla \varsigma - \varsigma \nabla_H \cdot \vec{V} + \vec{\omega}_H \cdot \nabla_H w \qquad (8)$$

The last term is called 'tilting' term. It turns horizontal vorticity into the vertical component through differential vertical motion.

Making use of Boussinesq mass continuity equation

$$\nabla \cdot \vec{V} = \nabla_H \cdot \vec{V} + \frac{\partial w}{\partial z} = 0$$

Eq.(8) can be rewritten as

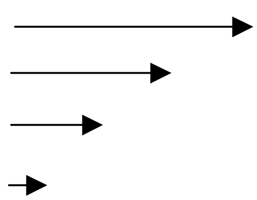
$$\frac{\partial \varsigma}{\partial t} = -\vec{V} \cdot \nabla \varsigma + \varsigma \frac{\partial w}{\partial z} + \vec{\omega}_H \cdot \nabla_H w \qquad (9)$$

Local change of vertical voricity	Advection term	Stretching term	Tilting term

Stretching term - if $\varsigma > 0$ and w increases with height (stretching of air column), the term is positive → increases in ς. Vertical stretching corresponds to horizontal convergence – angular momentum conservation principle!

Tilting term.

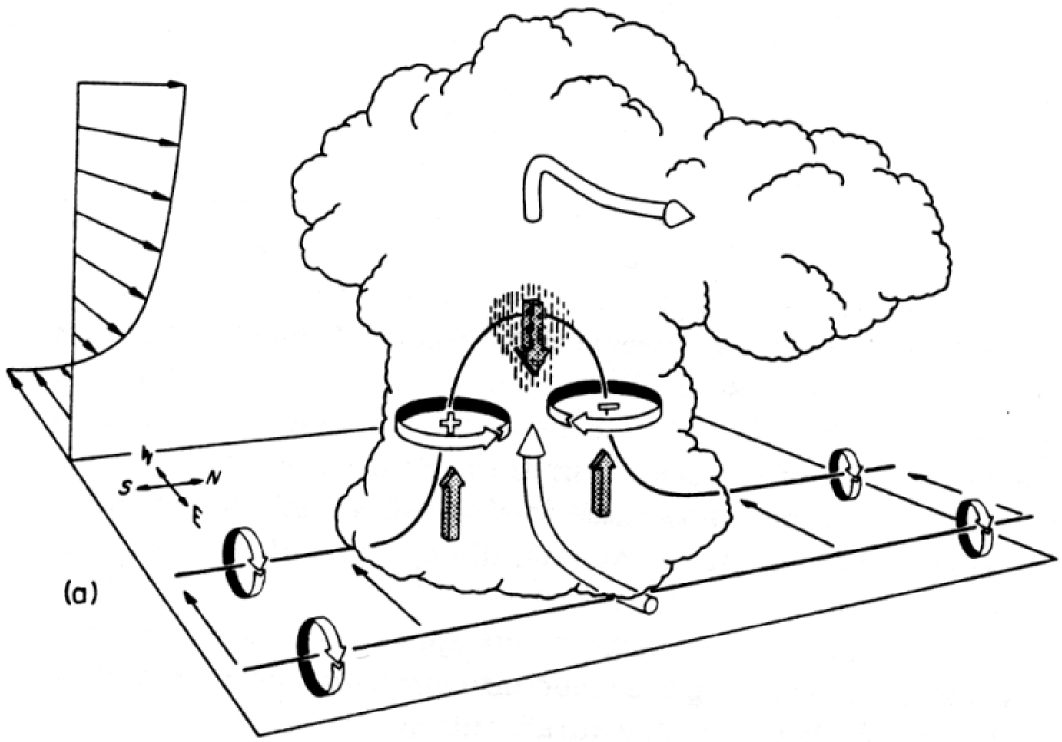
If shear is in u only,



$\frac{\partial u}{\partial z} > 0$, the horizontal vorticity vector points in positive y direction.

On the south side of updraft, $\frac{\partial w}{\partial y} > 0$ → $\frac{\partial \varsigma}{\partial t} = \eta \frac{\partial w}{\partial y} = \frac{\partial u}{\partial z}\frac{\partial w}{\partial y} > 0$ → the tilting creates positively vertical vorticity.

Similary, one the north side, the tilting due to updraft motion creates negative vertical voriticity.



2. Generation of Horizontal Vorticity (rotation about horizontal axis)

There the environment has vertical shear, the environment contains horizontal vorticity – the vertical shear is a reservoir of horizontal vorticity. We saw earlier that through tilting of horizontal vortex tubes, horizontal vorticity can be transformed into vertical one – contributing to the thunderstorm rotation.

We pointed earlier that buoyancy does not produce vertical vorticity, what about horizontal vorticity? Yes – it does. Remember horizontal vorticity generation at the gust front here where is strong horizontal buoyancy gradient?

To investigate generation of vorticity about the horizontal, we need the equation for horizontal vorticity. Let's consider the y component of vorticity first:

$$\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}.$$

Take $\frac{\partial}{\partial z}(Eq.1a) - \frac{\partial}{\partial x}(Eq.1c)$, we obtain

$$\boxed{\frac{d\eta}{dt} = -\frac{\partial B}{\partial x},} \qquad (10)$$

therefore (apart from friction), horizontal gradient of buoyancy provides the only source of horizontal vorticity processed by an air parcel.

When the low-level air parcel trajectory has a significant parallel component to the gust front, the vorticity generation by the horizontal gradient can be significant, and the tilting of it into the vertical is believed to contribute significantly to the thunderstorm rotation too.

C. Pressure Perturbation Equation

The rotational dynamics with supercell storms has a lot to do with the pressure perturbations created by the air flow. It is this effect that makes supercells special.

Take $\nabla \cdot$(equation of motion), i.e., $\frac{\partial}{\partial x}(1a) + \frac{\partial}{\partial y}(1c) + \frac{\partial}{\partial z}(1d)$:

1sr term: $$\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right] = -\frac{1}{\rho_0}\frac{\partial^2 p'}{\partial x^2}$$

2nd term: $$\frac{\partial}{\partial y}\left[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right] = -\frac{1}{\rho_0}\frac{\partial^2 p'}{\partial y^2}$$

3rd term: $$\frac{\partial}{\partial z}\left[\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right] = -\frac{1}{\rho_0}\frac{\partial^2 p'}{\partial z^2} + \frac{\partial B}{\partial z}.$$

Taking derivatives, combining terms and remembering $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$, we have

$$\frac{1}{\rho_0}\nabla^2 p' = -\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] - 2\left[\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial w}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial w}{\partial y}\right] + \frac{\partial B}{\partial z} \qquad (11)$$

Now let's partition winds between environmental wind and thunderstorm induced perturbation winds:

$$u(x,y,z,t) = \overline{u}(z) + u'(x,y,z,t)$$
$$v(x,y,z,t) = \overline{v}(z) + v'(x,y,z,t)$$
$$w(x,y,z,t) = 0 + w'(x,y,z,t)$$

Substituting them into (11), we have

$$\nabla^2 p' = -\rho_0\left[\left(\frac{\partial u'}{\partial x}\right)^2 + \left(\frac{\partial v'}{\partial y}\right)^2 + \left(\frac{\partial w'}{\partial z}\right)^2 + 2\frac{\partial u'}{\partial y}\frac{\partial v'}{\partial x} + 2\frac{\partial u'}{\partial z}\frac{\partial w'}{\partial x} + 2\frac{\partial v'}{\partial z}\frac{\partial w'}{\partial y}\right] - \rho_0\left[2\frac{\partial \overline{u}}{\partial z}\frac{\partial w'}{\partial x} + \frac{\partial \overline{v}}{\partial z}\frac{\partial w'}{\partial y}\right] + \rho_0\frac{\partial B}{\partial z} \qquad (12)$$

It's an elliptic diagnostic equation for pressure p'.

Dividing the total perturbation pressure into

$$p' = p'_{dyn} + p'_B$$
$$= p'_L \qquad + \qquad p'_{NL} \qquad + \qquad p'_B$$
$$2nd\ term \qquad "\ fluid\ extension\ term" + shear\ term \qquad last\ term$$

Each part is attributed to certain terms on the right hand side of Eq.(12).

Applications:

1. Updraft enhancement in rotating thunderstorms.

We saw earlier that a strong updraft in an environment of significant vertical shear produces a pair of counter-rotating vortices. Consider p'_{NL} term only. It can be verified that the 3 shear terms con be written in the following form:

$$2\frac{\partial u'}{\partial y}\frac{\partial v'}{\partial x}+2\frac{\partial u'}{\partial z}\frac{\partial w'}{\partial x}+2\frac{\partial v'}{\partial z}\frac{\partial w'}{\partial y}$$
$$=\frac{1}{2}\left[\left(\frac{\partial v'}{\partial x}+\frac{\partial u'}{\partial y}\right)^2+\left(\frac{\partial v'}{\partial z}+\frac{\partial w'}{\partial y}\right)^2+\left(\frac{\partial u'}{\partial z}+\frac{\partial w'}{\partial x}\right)^2\right.$$
$$\left.-\left(\frac{\partial v'}{\partial x}-\frac{\partial u'}{\partial y}\right)^2-\left(\frac{\partial v'}{\partial z}-\frac{\partial w'}{\partial y}\right)^2-\left(\frac{\partial u'}{\partial z}-\frac{\partial w'}{\partial x}\right)^2\right]$$

If we assume pure rotation (no div, deformation) and ignore extension terms (i.e., look at the effect of rotation only), then

$$\nabla^2 p'_{NL}=\frac{\rho_0}{2}\left(\frac{\partial v'}{\partial x}-\frac{\partial u'}{\partial y}\right)^2=\frac{\rho_0}{2}\varsigma'^2 .$$

Since the left hand side is a Laplacian operator, $\nabla^2 p'_{NL}\propto -p'_{NL}$, therefore

$$p'_{NL}\propto -\varsigma'^2 . \qquad (13)$$

→ Both cyclonic and anticyclonic rotation produces negative pressure perturbation. The low pressure center is actually required to have a PGF that balances the centrifugal force!

Negative p'_{NL} largest where rotation strongest, which is usually at the mid-levels of thunderstorms -- p'_{NL} "low" there up to 2-4 mb.

Earlier figure shows that because of tilting, vertical rotation is strongest at the flanks of updraft, and negative p' at the mid-levels creates a updraft pressure gradient force there that promotes new updrafts there – **a dynamic cause for cell splitting**.