

Lecture 13

Scattering. Part 1: Main concepts: scattering, elementary waves, polarization, Stokes matrix, and scattering phase function.

Rayleigh scattering.

Objectives:

1. Main concepts: scattering, elementary wave and light beam, polarization, Stoke matrix, and scattering phase function.
2. Rayleigh scattering.

Required Reading:

L80: 1.1.4; 3.6, 3.7

Additional/advanced Reading:

Le93: 12.1; 14/G&Y: 7.1

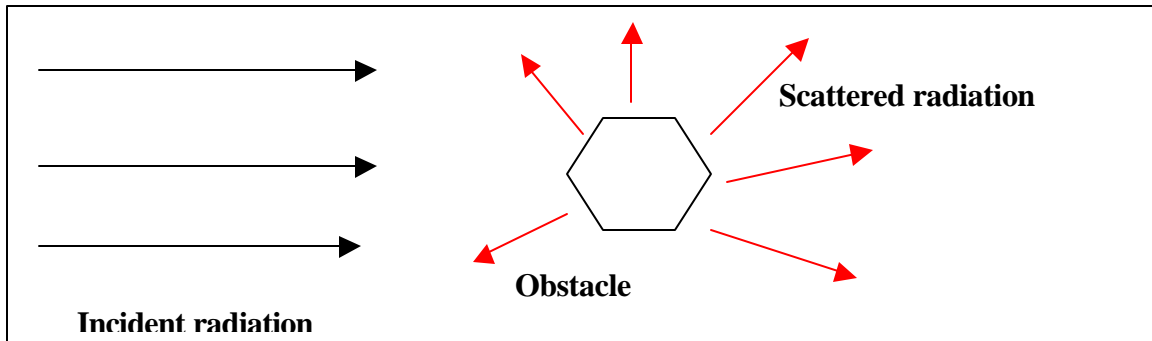
1. Main concepts: scattering, elementary wave and light beam, polarization, Stoke matrix, and scattering phase function.

Recall Lecture 2 where we have introduced the concepts of electromagnetic field.

- Scattering of the electromagnetic radiation is described by **the classical electromagnetic theory**, considering the propagation of a light beam as a transverse wave motion (collection of **electromagnetic individual waves**).

NOTE: The classical theory does treat absorption and emission processes. But the classical theory cannot provide the absorption cross sections of individual molecules (gases), which require the quantum theory (see Lecture 5).

Underlying physics of scattering:



Matter is composed of discrete electric charges: electrons and protons. If an obstacle (e.g., a single electron, atom, molecule, or particle) is illuminated by the electromagnetic wave, the electric charges in the obstacle oscillate radiating the “secondary” electromagnetic radiation in all directions. This is called **scattered radiation**.

Scattering can be considered as two step process: (1) excitation and (2) reradiation.

NOTE: In addition to scattering, the excited charges may transform some part of incident radiative energy into other forms of energy. This process is called **absorption**. *Thus, scattering and absorption are not mutually independent.*

- In our course we study only the case when the scattered radiation has the same frequency as that of the incident field (called **elastic scattering**).
- Electromagnetic field is characterized by the **electric vector** \vec{E} and **magnetic vector** \vec{H} , which are orthogonal to each other and to the direction of the propagation.

Poynting vector gives the flow of radiant energy and the direction of propagation as

$$\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{H} \quad [13.1]$$

in (cgs) unit system. \vec{E} is in units of charge per unit area, and S is in units of energy per unit time per unit area (i.e. flux). \vec{E} and \vec{H} obey the **Maxwell equations**.

NOTE: $\vec{E} \times \vec{H}$ means a **vector product** of two vectors.

If there are no free charges and no electrical current, **Maxwell equations** become

$$\begin{aligned}\nabla \times \vec{H} &= \frac{\mathbf{e}}{c} \frac{\partial \vec{E}}{\partial t} \\ \nabla \times \vec{E} &= -\frac{\mathbf{m}}{c} \frac{\partial \vec{H}}{\partial t} \\ \nabla \cdot \vec{E} &= 0 \\ \nabla \cdot \vec{H} &= 0\end{aligned}$$

where \mathbf{e} is the permittivity and \mathbf{m} if the permeability (both depend on the medium under consideration).

The velocity of light in the medium is

$$c = \frac{1}{\sqrt{\mathbf{em}}}$$

If the c_0 is the speed of light in vacuum, the **refractive index** (or **optical constants**) of the medium is defined as

$$m = c_0 \sqrt{\mathbf{em}}$$

- **The refractive index** $m = m_r + im_i$ in the complex number. The imaginary part m_i of the refractive index determines the attenuation of the wave as it propagates through the medium; the real part m_r of the refractive index the velocity of propagation $= c / m_r$. The refractive index is a strong function of the wavelengths.

NOTE: The intensity is related to the electrical field vector in (cgs or Gaussian units) as

$$I = \frac{1}{\Delta\Omega} \frac{c}{4p} |E|^2 \quad [13.2]$$

Consider an **individual wave** propagating in z-direction.

NOTE: “same name”: individual wave=elementary wave=simple wave

The electric vector \vec{E} may be decomposed into the parallel E_l and perpendicular E_r components as

$$E_l = a_l \exp(-i\mathbf{d}_l) \exp(-ikz + i\mathbf{v}t) \quad [13.2a]$$

$$E_r = a_r \exp(-i\mathbf{d}_r) \exp(-ikz + i\mathbf{v}t) \quad [13.2b]$$

where a_l and a_r are the positive **amplitude** of the parallel E_l and perpendicular E_r components, respectively;

\mathbf{d}_l and \mathbf{d}_r are the positive **phases** of the parallel E_l and perpendicular E_r components, respectively;

k is the propagation (or wave) constant, $k = 2\mathbf{p}/\lambda$

\mathbf{w} is the circular frequency, $\mathbf{w} = kc = 2\mathbf{p}c/\lambda$

NOTE: Complex variables are used to simplify mathematical exposition. To describe wave motion, one needs to describe both its amplitude and the phase. It is because complex numbers are double numbers (i.e. have the real and imaginary parts) they are appropriate in describing the waves. In addition, complex numbers obey the laws of ordinary algebra, and in their exponential expressions they are easily integrated and differentiated. Recall that $\exp(ix) = \cos(x) + i \sin(x)$

Eq.[13.2] can be written in cosine representation as

$$E_l = a_l \cos(\mathbf{z} + \mathbf{d}_l)$$

$$E_r = a_r \cos(\mathbf{z} + \mathbf{d}_r)$$

where $\mathbf{z} = kz - \mathbf{v}t$ and $\zeta + \delta$ is called **phase**.

Then we have

$$E_l / a_l = \cos(\mathbf{z}) \cos(\mathbf{d}_l) - \sin(\mathbf{z}) \sin(\mathbf{d}_l)$$

$$E_r / a_r = \cos(\mathbf{z}) \cos(\mathbf{d}_r) - \sin(\mathbf{z}) \sin(\mathbf{d}_r)$$

Performing simple mathematical manipulations (see L80: Eqs.3.36-3.38), we obtain

$$(E_l / a_l)^2 + (E_r / a_r)^2 - 2(E_l / a_l)(E_r / a_r) \cos(\mathbf{d}) = \sin^2(\mathbf{d}) \quad [13.3]$$

where $\mathbf{d} = \mathbf{d}_r - \mathbf{d}_l$ is the **phase difference** (or **phase shift**).

Eq.[13.3] represents an ellipse => **elliptically polarized wave**

If $\mathbf{d} = m\pi$ ($m = 0, +1; +/-2\dots$), then $\sin(\mathbf{d}) = 0$ and Eq.[13.3] becomes

$$\left(\frac{E_l}{a_l} \pm \frac{E_r}{a_r} \right)^2 = 0 \quad \text{or} \quad \frac{E_l}{a_l} = \pm \frac{E_r}{a_r} \quad [13.4]$$

Eq.[13.4] represents two perpendicular lines => **linearly polarized wave**

If $\mathbf{d} = m\pi/2$ ($m = +/-1; +/-2\dots$) and $a_l = a_r = a$, Eq.[13.3] becomes

$$E_l^2 + E_r^2 = a^2 \quad [13.5]$$

Eq.[13.5] represents a circle => **circularly polarized wave**

- If there is a definite relation of phases between different scatterers => radiation is called **coherent**. If there is no relations in phase shift => light is called **incoherent**
- **Natural light is incoherent.**

The property of incoherent radiation:

The intensity due to all scattering centers is the sum of individual intensities.

NOTE: In our course, we study the **elastic incoherent scattering of the atmospheric radiation.**

NOTE: The assumption of independent scatterers is violated if the particles are too closely packed (spacing between particles should be several times their diameters to prevent intermolecular forces from causing correlation between scattering centers).

- Eq.[13.3] shows that, in the general case, three independent parameters a_l , a_r and \mathbf{d} are required to characterize an electromagnetic wave. These parameters are not measured. Therefore, a new set of parameters (which are proportional to intensity) has been proposed by Stoke.

Stokes parameters: so-called intensity I, the degree of polarization Q, the plane of polarization U, and the ellipticity V of the electromagnetic wave.

$$\begin{aligned}
 I &= E_l E_l^* + E_r E_r^* \\
 Q &= E_l E_l^* - E_r E_r^* \\
 U &= E_l E_r^* + E_r E_l^* \\
 V &= -i(E_l E_r^* - E_r E_l^*)
 \end{aligned}
 \tag{13.6}$$

They are related as

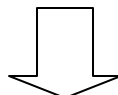
$$I^2 = Q^2 + U^2 + V^2 \tag{13.7}$$

Stokes parameter can be also expressed as

$$\begin{aligned}
 I &= a_l^2 + a_r^2 \\
 Q &= a_l^2 - a_r^2 \\
 U &= 2a_l a_r \cos(\mathbf{d}) \\
 V &= 2a_l a_r \sin(\mathbf{d})
 \end{aligned}
 \tag{13.8}$$

- Actual light consists of **many individual waves** each having its own amplitude and phase.

NOTE: During a second, a detector collects about millions of individual waves.



Measurable intensities are associated with the superposition of many millions of simple waves with independent phases. Therefore, for a light beam the Stokes parameters are averaged over a time period and may be represented as

$$\begin{aligned}
 I &= \langle a_l^2 \rangle + \langle a_r^2 \rangle = I_l + I_r \\
 Q &= \langle a_l^2 \rangle - \langle a_r^2 \rangle = I_l - I_r \\
 U &= \langle 2a_l a_r \cos(\mathbf{d}) \rangle \\
 V &= \langle 2a_l a_r \sin(\mathbf{d}) \rangle
 \end{aligned}
 \tag{13.9}$$

where $\langle \dots \rangle$ denote the time averaging.

It can be shown that for a light beam we have

$$I^2 \geq Q^2 + U^2 + V^2$$

The **degree of polarization** P of a light beam is defined as

$$P = (Q^2 + U^2 + V^2)^{1/2} / I$$

The **degree of linear polarization** LP of a light beam is defined by neglecting U and V as

$$LP = \frac{Q}{I} = \frac{I_l - I_r}{I_l + I_r} \tag{13.10}$$

The **phase function $P(\cos\Theta)$** is defined as a non-dimensional parameter to describe the angular distribution of the scattered radiation as

$$\frac{1}{4\pi} \int_{\Omega} P(\cos\Theta) d\Omega = 1 \quad [13.11]$$

where Θ is called the **scattering angle** between the direction of incidence and observation.

NOTE: The phase function is expressed as

$$P(\cos\mathbf{Q}) = P(\mathbf{q}', \mathbf{f}', \mathbf{q}, \mathbf{f}),$$

where $(\mathbf{q}', \mathbf{f}')$ and (\mathbf{q}, \mathbf{f}) are the spherical coordinates of incident beam and direction of observation.

Using the spherical geometry it can be shown (see L80: Appendix F) that

$$\cos(\mathbf{Q}) = \cos(\mathbf{q}')\cos(\mathbf{q}) + \sin(\mathbf{q}')\sin(\mathbf{q}) \cos(\mathbf{f}'-\mathbf{f}),$$

Forward scattering refers to the observations directions for which $\Theta < \pi/2$

Backward scattering refers to the observations directions for which $\Theta > \pi/2$

Plane of scattering (or **scattering plane**) is defined as a plane containing the incident beam and scattered beam in the direction of observation.

- Particles of different sizes, shapes and indices of refraction will have different scattering properties.

Scattering domains

Rayleigh scattering: $2\pi r/l \ll 1$, m is arbitrary (applies to scattering by molecules and small aerosol particles)

Rayleigh-Gans scattering: $(m-1) \ll 1$ (not useful for atmospheric application)

Mie-Debye scattering: $2\pi r/l$ and m are both arbitrary but for spheres only (applies to scattering by aerosol and cloud particles)

Geometrical optics: $2\pi r/l$ is very large and m is real (applies to scattering by large cloud droplets).

2. Rayleigh scattering

- Because the sizes of atmospheric molecules are much smaller than the wavelengths of solar and IR radiation, the scattering by atmospheric gases is the Rayleigh scattering.

Consider small homogeneous spherical particle (e.g., molecule) with size smaller than the wavelength of incident radiation \vec{E}_0 . Let \vec{p}_0 be the induced dipole moment, then from the classical electromagnetic theory we have

$$\vec{p}_0 = \mathbf{a}\vec{E}_0$$

where \mathbf{a} is the polarizability of the particle.

According to the classical electromagnetic theory, the scattered electric field at the large distance r (called far field scattering) from the dipole is given (in cgs units) by

$$\vec{E} = \frac{1}{c^2} \frac{1}{r} \frac{\partial \vec{p}}{\partial t} \sin(\mathbf{g})$$

where \mathbf{g} is the angle between the scattered dipole moment \vec{p} and the direction of observation.

In oscillating periodic field, the dipole moment is given in terms of induced dipole moment by

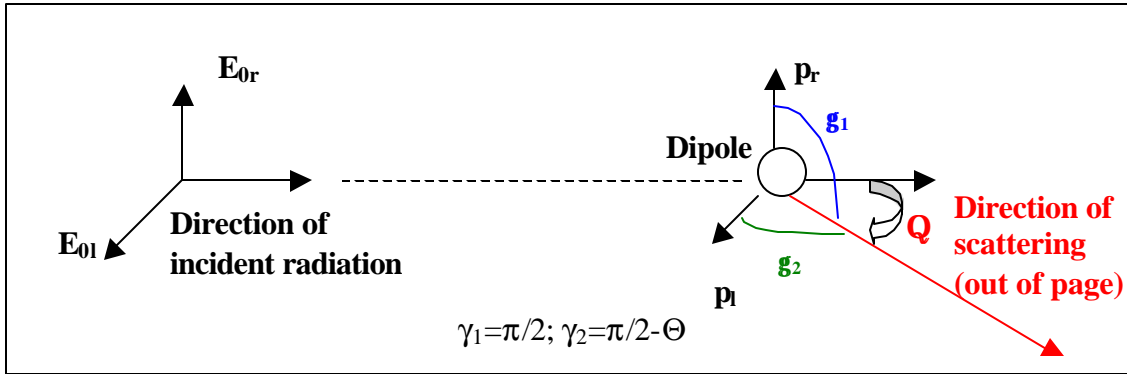
$$\vec{p} = \vec{p}_0 \exp(-ik(r - ct))$$

and thus the electrical field is

$$\vec{E} = -\vec{E}_0 \frac{\exp(-ik(r - ct))}{r} k^2 \mathbf{a} \sin(\mathbf{g})$$

Decomposing the electrical vector on two orthogonal components perpendicular and parallel to the plane of scattering (a plane containing the incident and scattering beams), We have

$$E_r = -E_{0r} \frac{\exp(-ik(r - ct))}{r} k^2 \mathbf{a} \sin(\mathbf{g}_1)$$
$$E_t = -E_{0t} \frac{\exp(-ik(r - ct))}{r} k^2 \mathbf{a} \sin(\mathbf{g}_2)$$



Using that

$$I = \frac{1}{\Delta\Omega} \frac{c}{4\pi} |E|^2,$$

perpendicular and parallel intensities (or linear polarized intensities) are

$$I_r = I_{0r} k^4 a^2 / r^2$$

$$I_l = I_{0l} k^4 a^2 \cos^2(\Theta) / r^2$$

Using that the natural light (incident beam) is not polarized ($I_{0r} = I_{0l} = I_0/2$) and that

$k = 2\pi/\lambda$, we have

$$I = I_r + I_l = \frac{I_0}{r^2} a^2 \left(\frac{2\pi}{\lambda} \right)^4 \frac{1 + \cos^2(\Theta)}{2} \quad [13.12]$$

Eq.[13.12] gives the intensity scattered by molecules for unpolarized incident light, Rayleigh scattering.

Rayleigh scattering phase function for incident unpolarized radiation (follows from

Eq.[13.12]) is

$$P(\cos(\Theta)) = \frac{3}{4} (1 + \cos^2(\Theta)) \quad [13.13]$$

Eq.[13.12] may be rewritten in the form

$$I(\cos(\Theta)) = \frac{I_0}{r^2} \mathbf{a}^2 \frac{128\mathbf{p}^5}{3\mathbf{I}^4} \frac{P(\Theta)}{4\mathbf{p}} \quad [13.14]$$

NOTE: Angular distribution of the scattered intensity is associated with the phase function.

- Rayleigh scattering results in the sky polarization. The degree of linear polarization is

$$LP(\Theta) = -\frac{Q}{I} = -\frac{I_l - I_r}{I_l + I_r} = \frac{\cos^2 \Theta - 1}{\cos^2 \Theta + 1} = \frac{\sin^2 \Theta}{\cos^2 \Theta + 1}$$

Eq.[13.12] may be rewritten in the terms of the scattering cross section

$$I(\cos(\Theta)) = \frac{I_0}{r^2} \mathbf{s}_s \frac{P(\Theta)}{4\mathbf{p}} \quad [13.15]$$

Here the scattering cross section (in units of area) by a single molecule is

$$\mathbf{s}_s = \mathbf{a}^2 \frac{128\mathbf{p}^5}{3\mathbf{I}^4}$$

The polarizability \mathbf{a} is given by the Lorentz-Lorenz formula (see the derivation in L80: Appendix D):

$$\mathbf{a} = \frac{3}{4\mathbf{p}N} \left(\frac{m^2 - 1}{m^2 + 2} \right)$$

where N is the number of molecules per unit volume and $m = m_r + im_i$ is the refractive index.

- For air molecules in solar spectrum:
 m_r is about 1 but strongly depends on λ , and $m_i = 0$.

Thus the polarizability can be approximated as

$$\mathbf{a} \approx \frac{1}{4\mathbf{p}N} (m_r^2 - 1)$$

Therefore, the scattering cross section of air molecules becomes

$$\mathbf{s}_s = \frac{8\mathbf{p}^3 (m_r^2 - 1)^2}{3\mathbf{I}^4 N^2} f(\mathbf{d}) \quad [13.16]$$

where $f(\mathbf{d})$ is the correction factor for the anisotropic properties of air molecules, defined as $f(\mathbf{d}) = (6+3\mathbf{d})/(6-7\mathbf{d})$ and $\mathbf{d}=0.035$

Using this scattering cross section, one can calculate the optical depth of the entire atmosphere due to molecular scattering as

$$\mathbf{t}(\mathbf{I}) = \mathbf{s}_s(\mathbf{I}) \int_0^{\text{top}} N(z) dz$$