Lecture 18

Methods for solving the radiative transfer equation. Part 1: Two-stream approximations.

Objectives:

1. Concepts of the reflection and transmission of an atmospheric layer.
2. Two-stream approximations.
3. Eddington and Delta-Eddington approximations.

Required reading:

L80: 6.2; 6.4.1

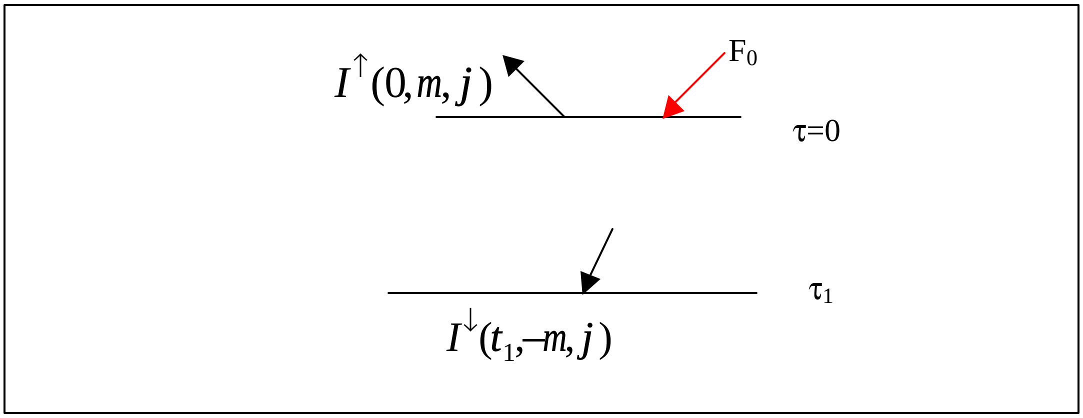
For Homework 3: King M.D. and Harshvardhan "Comparative accuracy of selected multiple scattering approximations" Journal of the Atmospheric Sciences 43, 784-801, 1986.

Advanced reading:

G&Y: 2.4.4-2.4.5; 8.4.5

1. Concepts of the reflection and transmission of an atmospheric layer.

Consider the atmosphere with an optical depth $\mathbf{t}_1$.



$I^{\uparrow}(0, m, j)$ can be considered as the reflected diffuse intensity

$I^{\downarrow}(t_1, -m, j)$ can be considered as transmitted diffuse intensity

Reflection function of an atmospheric layer is defined as

$$R(m, j, m_0, j_0) = I^{\uparrow}(0, m, j) / m_0 F_0 \qquad [18.1]$$

Transmission function of an atmospheric layer is defined as

$$T(m, j, m_0, j_0) = I^{\downarrow}(t_1, -m, j) / m_0 F_0 \qquad [18.2]$$

NOTE: Eq.[18.2] uses the diffuse intensity, therefore $T(m, j, m_0, j_0)$ is also called the diffuse transmission function.

Transmission function for direct solar radiation is defined as

$$T_{dir}(m_0, j_0) = I_{dir}(t_1, -m_0, j_0) / m_0 F_0 = \exp(-t_1 / m_0) \qquad [18.3]$$

Planetary albedo (or local albedo or reflection) is associated with the reflected (upward) flux and defined as

$$r(m_0) = \frac{F_{dif}^{\uparrow}(0)}{p m_0 F_0} = \frac{1}{p} \int_0^{2p} \int_0^1 R(m, j, m_0, j_0) m\, dm\, dj \qquad [18.4]$$

Diffuse transmission is associated with transmitted (downward) flux and defined as

$$t(m_0) = \frac{F_{dif}^{\downarrow}(t_1)}{p m_0 F_0} = \frac{1}{p} \int_0^{2p} \int_0^1 T(m, j, m_0, j_0) m\, dm\, dj \qquad [18.5]$$

For the azimuthally independent case, Eqs.[18.4]-[18.5] reduce to

$$r(m_0) = 2 \int_0^1 R(m, m_0) m\, dm \qquad [18.6]$$

$$t(m_0) = 2 \int_0^1 T(m, m_0) m\, dm \qquad [18.7]$$

Consider a planet of radius a. The total amount of energy per unit time is

$$pa^2pF_0$$

Spherical (or global) albedo is a ratio of the energy reflected by the entire planet to the energy incident on it and defined as

$$\bar{r} = \frac{f^{\uparrow}(0)}{pa^2pF_0} = 2\int_0^1 r(m_0)m_0dm_0 \qquad [18.8]$$

Global diffuse transmission is defined as

$$\bar{t} = \frac{f^{\downarrow}(t_1)}{pa^2pF_0} = 2\int_0^1 t(m_0)m_0dm_0 \qquad [18.9]$$

2. Two-stream approximations.

Underlying idea:

Because radiation flux and heating rates are angular-averaged properties, one can expect that details of the angular variation of intensity are not very important for the predictions of these quantities.

Strategy:

Introduce an “effective” angular averaged intensity/flux (stream) so the integro-differential equation of radiative transfer (e.g., Eq.[17.6]) reduces to two coupled ordinary differential equations.

Disadvantages of the two-stream approximations:

Two-steam methods provide acceptable accuracy but over a restricted range of the parameters. There is no a priory method to estimate the accuracy, so one needs to use the “exact” method to obtain an accurate solution which can be used to estimate the accuracy of two-stream solutions.

Advantages of the two-stream approximations:

Two-stream approximations are computationally efficient (therefore they are often used in climate models) and often sufficiently accurate.

- To implement a two-stream technique, one must decide on how to determine the “effective” intensity (i.e., the effective scattering angle).

One possible strategy: define $\overline{m}^{\uparrow\downarrow}$ as the intensity-weighted angular means

$$\overline{m}^{\uparrow\downarrow} = \frac{\int_0^1 I^{\uparrow\downarrow}(t,m)\,m\,dm}{\int_0^1 I^{\uparrow\downarrow}(t,m)\,dm} \qquad [18.10]$$

For **isotropic radiation filed**, Eq.[18.10] gives $\overline{m}^{\uparrow\downarrow} = 1/2$

Another possible strategy: define $\overline{m}^{\uparrow\downarrow}$ as the root-mean square value

$$\overline{m}^{\uparrow\downarrow} = \sqrt{\langle m^2 \rangle} = \sqrt{\frac{\int_0^1 I^{\uparrow\downarrow}(t,m)\,m^2\,dm}{\int_0^1 I^{\uparrow\downarrow}(t,m)\,dm}} \qquad [18.11]$$

For **isotropic radiation filed**, Eq.[18.11] gives $\overline{m}^{\uparrow\downarrow} = 1/\sqrt{3}$

NOTE: A problem with these two approaches (Eqs.[18.10] and [18.11]) is that we don’t know a priori the angular distribution on the intensity.

⇩

A better strategy is to utilize the Gaussian quadrature

Gauss quadrature applied to any function ***f*****(m)** gives

$$\int_{-1}^{1} f(m)dm \approx \sum_{j=-n}^{n} a_j f(m_j) \qquad [18.12]$$

where a_j are the weights defined as

$$a_j = \frac{1}{P_{2n}{}'(m_j)} \int_{-1}^{1} \frac{P_{2n}(m)}{m - m_j} dm \qquad [18.13]$$

and m_j are the zeros of the even-order Legendre polynomials $P_{2n}(m)$

NOTE: see table 6.1 in L80 for Gaussian points and weights for the first four approximations (i.e., for ***n***=4)

Recall the equation of the radiative transfer for the diffuse intensity Eq.[17.17] for the azimuth-independent case

$$m\frac{dI(t,m)}{dt} = I(t,m) - \frac{v_0}{2}\sum_{l=0}^{N} v_l^* P_l(m)\int_{-1}^{1} P_l(m')I(t,m')dm' -$$

$$-\frac{v_0}{4p}\sum_{l=0}^{N} v_l^* P_l(m) P_l(-m_o) pF_0 \exp(-t/m_0) \qquad [17.17]$$

Using Gaussian quadrature, we can re-write the radiative transfer equation as

$$m_i\frac{dI(t,m_i)}{dt} = I(t,m_i) - \frac{v_0}{2}\sum_{l=0}^{N} v_l^* P_l(m_i)\sum_{j=-n}^{n} a_j P_l(m_j) I(t,m_j) -$$

$$-\frac{v_0}{4p}[\sum_{l=0}^{N}(-1)^l v_l^* P_l(m_i) P_l(-m_o)] pF_0 \exp(-t/m_0)$$

[18.14]

where ***i = -n, n*** and m_i***(-n, n)*** represent the directions of radiation streams.

For ***n=1*** (and hence ***j*** = -1 and 1 or two streams), we have $m_1 = \frac{1}{\sqrt{3}}$ and $\boldsymbol{a_j} = \boldsymbol{a_{-j}} = 1$

For this case, Eq.[18.14] splits into two equations

$$m_1 \frac{dI^{\uparrow}(t,m_1)}{dt} = I^{\uparrow}(t,m_1) - v_0(1-b)I^{\uparrow}(t,m_1) - v_0 b I^{\downarrow}(t,-m_1) - S^{-}\exp(-t/m_0)$$

[18.15a]

$$-m_1 \frac{dI^{\downarrow}(t,-m_1)}{dt} = I^{\downarrow}(t,-m_1) - v_0(1-b)I^{\downarrow}(t,-m_1) - v_0 b I^{\uparrow}(t,m_1) - S^{+}\exp(-t/m_0)$$

[18.15b]

where

$$S^{\pm} = \frac{F_0 v_0}{4}(1 \pm 3 g m_1 m_0)$$

$g = \frac{v_1^*}{3} = \frac{1}{2}\int_{-1}^{1} P(\cos\Theta)\cos(\Theta)d\cos(\Theta)$, ***g*** is the asymmetry parameter.

$b = \frac{1-g}{2} = \frac{1}{2}\int_{-1}^{1} P(\cos\Theta)\frac{1-\cos(\Theta)}{2}d\cos(\Theta)$; ***b*** can be interpreted as a backscattered fraction of energy and ***(1-b)*** is forward scattered energy.

The solutions of Eqs.[18.15a,b] are

$$I^{\uparrow} = I(t, m_1) = Kv\exp(kt) + Hu\exp(-kt) + e\exp(-t/m_0)$$

[18.16a]

$$I^{\downarrow} = I(t,-m_1) = Ku\exp(kt) + Hv\exp(-kt) + g\exp(-t/m_0)$$

[18.16b]

where

$$v = (1 + a)/2\,; \qquad u = (1 - a)/2$$

$$a^2 = \frac{1 - v_0}{1 - g v_0}\,; \quad k^2 = \frac{(1 - v_0)(1 - g v_0)}{m_1^2}$$

$$e = \frac{a + b}{2}\,; \qquad g = \frac{a - b}{2}$$

$$a = \frac{Z_1 m_0^2}{1 - m_0^2 k^2}\,; \qquad b = \frac{Z_2 m_0^2}{1 - m_0^2 k^2}$$

$$Z_1 = -\frac{(1 - g v_0)(S^- + S^+)}{m_1^2} + \frac{S^- + S^+}{m_1 m_0}\,; \; Z_2 = -\frac{(1 - v_0)(S^- - S^+)}{m_1^2} + \frac{S^- + S^+}{m_1 m_0}$$

The constant ***K*** and ***H*** are determined from the boundary conditions on the top and at the bottom of the atmospheric layer. Using the boundary conditions given by Eq.[17.8] (i.e., no diffuse downward radiation at the top of the atmosphere and no reflection from the surface), we have

$$K = -\frac{ev \exp(-t_1/m_0) - gu \exp(-kt_1)}{v^2 \exp(kt_1) - u^2 \exp(-kt_1)}$$

$$H = -\frac{eu \exp(-t_1/m_0) - gv \exp(-kt_1)}{v^2 \exp(kt_1) - u^2 \exp(-kt_1)}$$

From the upward and downward intensities we can write expressions for **upward downward diffuse fluxes** in the two-stream approximations:

$$\boxed{\begin{aligned} F^{\uparrow}(t) &= 2pm_1 I^{\uparrow}(t, m_1) \\ F^{\downarrow}(t) &= 2pm_1 I^{\downarrow}(t, -m_1) \end{aligned}}$$

[18.17a]

[18.17b]

3. Eddington and Delta-Eddington approximations.

Strategy of the Eddington approximation:

Approximate the angular dependence of the diffuse intensity by a polynomial in μ.

Choosing a linear polynomial gives

$$I(t,m) = I_0(t) + I_1(t)m \;;\quad -1 \le m \le 1 \qquad [18.18]$$

Substituting this approximation into the radiative transfer equation, we have

$$\boxed{\begin{aligned} \frac{dI_1}{dt} &= 3(1-v_0)I_0 - \frac{3}{4}v_0 F_0 \exp(-t/m_o) && [18.19a] \\ \frac{dI_0}{dt} &= 3(1-v_0 g)I_1 + \frac{3}{4}v_0 g m_0 F_0 \exp(-t/m_o) && [18.19b] \end{aligned}}$$

Diffuse fluxes for the Eddington approximation become

$$F^{\uparrow}(t) = 2p\int_0^1 \left[I_0(t) + mI_1(t)\right] m\,dm = p\left[I_0(t) + \frac{3}{2}I_1(t)\right] \qquad [18.20a]$$

$$F^{\downarrow}(t) = 2p\int_0^{-1} \left[I_0(t) + mI_1(t)\right] m\,dm = p\left[I_0(t) - \frac{3}{2}I_1(t)\right] \qquad [18.20b]$$

- The two-stream and Eddington methods are good approximations for optically thick layer, but they may produce inaccurate results for thin layers and strong absorption. The main problem is that the phase function is highly peaked in the forward direction.

Delta-function adjustment is introduced to incorporate the forward peak contribution by adjusting optical properties such that the fraction of scattered energy in the forward direction, **f,** is removed from the scattering parameters

$$g' = \frac{g - f}{1 - f} \qquad v'_o = \frac{(1 - f)v_0}{1 - fv_0} \qquad t' = (1 - fv_0)t$$

The phase function may be expressed as

$$P(m, m') = 2fd(m - m') + (1 - f)(1 + 3g'mm') \qquad [18.21]$$

In the limit of the Henyey-Greenstein phase function: $f = g^2$

- The incorporation of the δ-function adjustment into two-stream and Eddington methods greatly improves their accuracy.

NOTE: King and Harshvardhan (1986) performed analysis of the accuracy of two-stream and **d**-two-stream approximations, and Eddington and δ-Eddington approximations.