Lecture 20

Methods for solving the radiative transfer equation. Part 3: Discrete-ordinate method.

Objectives:

1. D-O method for the case of isotropic scattering.
2. Generalization of the D-O method for inhomogeneous atmosphere.

Required reading:

L80: 6.3;

Recommended reading

Thomas G.E. and K. Stamnes, Radiative transfer in the atmosphere and ocean, 2000, Chapter 8.

1. D-O method for the case of isotropic scattering.

- A discrete-ordinate (D-O) method has been developed by Chandrasekhar in about 1950 (see Chandrasekhar S., Radiative transfer, 1960, Dover Publications).

Recall the radiative transfer equation Eq.[17.6] for azimuthally independent diffuse intensity:

$$m \frac{dI(t,m)}{dt} = I(t,m) - \frac{v_0}{2}\int_{-1}^{1} I(t,m')P(m,m')dm' -$$
$$- \frac{v_0}{4p} pF_0 P(m,-m_0)\exp(-t/m_0) \qquad [17.6]$$

For isotropic scattering, the scattering phase function is 1. Hence we have

$$m \frac{dI(t,m)}{dt} = I(t,m) - \frac{v_0}{2}\int_{-1}^{1} I(t,m')dm' - \frac{v_0 F_0}{4}\exp(-t/m_0) \qquad [20.1]$$

Let's apply the Gauss formula (see Eq.[18.12]) to replace the integral in Eq.[20.1]

$$m_i \frac{dI(t,m_i)}{dt} = I(t,m_i) - \frac{v_0}{2}\sum_{j=-n}^{n} a_j I(t,m_j) - \boxed{\frac{v_0 F_0}{4}\exp(-t/m_0) \quad \text{Inhomogeneous part}} \qquad [20.2]$$

where $i=-n, \ldots, n$ (2n terms) and $\boldsymbol{a_j}$ are the Gaussian weights (constants) and $\boldsymbol{m_j}$ are quadrature angles (or points).

Eq.[20.2] is a system of 2n inhomogeneous differential equations:

Solution of Eq.[20.2] = general solution + particular solution

General solution is a solution of the homogeneous part of the Eq.[20.2]

Denoting $\boldsymbol{I_i = I_i(t, m_i)}$, the general solution of Eq.[20.2] can be found as

$$I_i = g_i \exp(-kt) \qquad [20.3]$$

Inserting Eq.[20.3] into Eq.[20.2], we obtain

$$g_i(1+m_i k) = \frac{v_0}{2}\sum_{j=-n}^{n} a_j g_j \qquad [20.4]$$

We can find $\boldsymbol{g_i}$ in the form

$$g_i = L/(1+m_i k)$$

where L is a constant to be determined. Substituting this expression for $\boldsymbol{g_i}$ in Eq.[20.4], we have

$$1 = \frac{v_0}{2}\sum_{j=-n}^{n} \frac{a_j}{1+m_i k} = v_0 \sum_{j=1}^{n} \frac{a_j}{1-m_j^2 k^2} \qquad [20.5]$$

Eq.[20.5] gives 2n solutions for +/k$_j$ (j=1,…,n).

Thus general solution is

$$I_i = \sum_j \frac{L_j}{1+m_i k_j}\exp(-k_j t) \qquad [20.6]$$

where $\boldsymbol{L_j}$ are constants.

The particular solution can be found as

$$I_i = \frac{v_0 F_0}{4} h_i \exp(-t / m_0) \qquad [20.7]$$

where $\boldsymbol{h_i}$ are constants.

Inserting Eq.[20.7] into Eq.[20.2], we have

$$h_i (1 + m_i / m_0) = \frac{v_0}{2} \sum_{j=-n}^{n} a_j h_j + 1 \qquad [20.8]$$

From Eq.[20.8], $\boldsymbol{h_i}$ is found as

$$h_i = g / (1 + m_i / m_0)$$

where γ is determined from

$$g = 1/\{1 - \frac{v_0}{2} \sum_{j=1}^{n} a_j / (1 - m_j^2 / m_0^2)\} \qquad [20.9]$$

Adding the general solution Eq.[20.6] and the particular solution Eq.[20.7], we have the solution

$$I_i = \sum_j \frac{L_j}{1 + m_i k_j} \exp(-k_j t) + \frac{v_0 F_0 g}{4(1 + m_i / m_0)} \exp(-t / m_0) \qquad [20.10]$$

where $\boldsymbol{L_j}$ are constants to be determined from the boundary conditions.

H-function has been introduced by Chandrasekhar as

$$H(m) = \frac{1}{m_1 \ldots m_n} \frac{\prod_{j=1}^{n} (m + m_j)}{\prod_{j=1}^{1} (1 + k_j m)} \qquad [20.11]$$

One can express **g** in the H-function that Eq.[20.10] becomes

$$I_i = \sum_j \frac{L_j}{1 + m_i k_j} \exp(-k_j t) + \frac{v_0 F_0 H(m_0) H(-m_0)}{4(1 + m_i / m_0)} \exp(-t / m_0)$$

[20.12]

Eq.[20.12] gives a simple solution for the semi-infinite isotropic atmosphere:

$$I^{\uparrow}(0,m)=\frac{1}{4}v_0F_0\frac{m_0}{m+m_0}H(m_0)H(m) \qquad [20.13]$$

2. Generalization of the D-O method for inhomogeneous atmosphere.

Let's consider the atmosphere with non-isotropic (anisotropic) scattering.

Recall that we may expand the diffuse intensity in the cosine series

$$I(t,m,j)=\sum_{m=0}^{N}I^{m}(t,m)\cos(m(j_0-j)) \qquad [17.14]$$

So we need to solve Eq.[17.15]

$$m\frac{dI^m(t,m)}{dt}=I^m(t,m)-(1+d_{0,m})\frac{v_0}{4}\sum_{l=m}^{N}v_l^mP_l^m(m)\int_{-1}^{1}P_l^m(m')I^m(t,m')dm'-$$

$$-\frac{v_0}{4p}\sum_{l=m}^{N}v_l^mP_l^m(m)P_l^m(-m_o)pF_0\exp(-t/m_0)$$

[17.15]

The **general solution** may be written

$$I^m(t,m_i)=\sum_{j=-n}^{n}L_j^mf_j^m(m_j)\exp(-k_j^mt)$$

f_j^m, k_j^m, L_j^m are coefficients to be determined.

The **particular solution** may be written

$$I_p^m(t,m_i)=Z^m(m_i)\exp(-t/m_0)$$

$Z^m(m_i)$ is a function

$$Z^m(m_i)=\frac{1}{4}v_0F_0P_m^m(-m_0)\frac{H^m(m_0)H^m(-m_0)}{1+m_i/m_0}\sum_{l=0}^{N}v_l^mz_l^m\frac{1}{m_0}P_l^m(m_i)$$

The **complete solution** of the radiative transfer is

$$I^m(t, m_i) = \sum_{j=-n}^{n} L_j^m f_j^m(m_j) \exp(-k_j^m t) + Z^m(m_i) \exp(-t / m_0) \qquad [20.14]$$

i=-n,…,n

Let's generalize the **complete solution** Eq.[20.14] of the radiative transfer for the inhomogeneous atmosphere. The atmosphere can be divided into the N homogeneous layers, each is characterized by a single scattering albedo, phase function, and optical depth.

NOTE: If a layer has gases, aerosols and/or clouds, one needs to calculate the effective optical properties of this layer.

For ***l***-th layer, we can write the solution using Eq.[20.14]. To simplify notations, let's consider the azimuthal independent case (i.e., ***m=0***), so we have

$$I^l(t, m_i) = \sum_{j=-n}^{n} L_j^l f_j^l(m_j) \exp(-k_j^l t) + Z^l(m_i) \exp(-t / m_0) \qquad [20.15]$$

Now, we need to match the boundary and continuity conditions between layers.

At the top of the atmosphere (TOA): no downward diffuse intensity

$$I^{l=1}(0, -m_i) = 0 \qquad [20.16]$$

At the layer's boundary: upward and downward intensities must be continuous

$$I^l(t_l, m_i) = I^{l+1}(t_l, m_i) \qquad [20.17]$$

At the bottom of the atmosphere (assuming the Lamdertian surface):

$$I^{l=N}(t_N, m_i) = \frac{r_{sur}}{p}[F^{\downarrow}(t_N) + m_0 F_0 \exp(-t_N / m_0)] \qquad [20.18]$$

Eqs.[20.16]-[20.18] provide necessary equations to find the unknown coefficients.

Few comments on DISORT

(DISORT is a numerical code based on the D-O method developed by Wiscombe et al.; see Laboratory 9)

1) DISORT applies to the inhomogeneous nonithothermal plane-parallel atmosphere.

2) A user may set-up any numbers of the plane-parallel layers.

3) Each layer must be characterized by the effective optical depth, single scattering albedo and asymmetry parameter if the Henyey-Greenstein phase function is used.

4) A user may use any phase function by providing the Legendre polynomial expansion coefficients.

5) A user selects a number of streams (keeping in mind that the computation time varies as n^3).

6) A key problem is to obtain a solution for fluxes for strongly forward-peaked scattering.

7) DISORT allows predicting the intensity as a function of the direction and position at any point in the atmosphere (i.e., not only at the boundaries of the layers).