Lecture 21

Methods for solving the radiative transfer equation. Part 4: Principles of invariance. Adding method.

Objectives:

1. Principles of invariance.
2. Adding method.

Required reading:

L80: 6.4-6.6

Recommended reading:

G&Y: 8.3

1. Principles of invariance.

Recall the definitions of reflection and transmission of a layer introduced in Lec.18-19.

If solar flux is incident on a layer of optical depth τ_1:

$$R(\mu,\varphi,\mu_0,\varphi_0) = I^{\uparrow}(0,\mu,\varphi)/\mu_0 F_0 \qquad [18.1]$$

$$T(\mu,\varphi,\mu_0,\varphi_0) = I^{\downarrow}(\tau_1,-\mu,\varphi)/\mu_0 F_0 \qquad [18.2]$$

General case:

$$I_r^{\uparrow}(0,\mu,\varphi) = \frac{1}{\pi}\int_0^{2\pi}\int_0^1 R(\mu,\varphi,\mu',\varphi')I_{inc}(-\mu',\varphi')\mu' d\mu' d\varphi' \qquad [19.4]$$

$$I_t^{\downarrow}(\tau_1,-\mu,\varphi) = \frac{1}{\pi}\int_0^{2\pi}\int_0^1 T(\mu,\varphi,\mu',\varphi')I_{inc}(-\mu',\varphi')\mu' d\mu' d\varphi' \qquad [19.5]$$

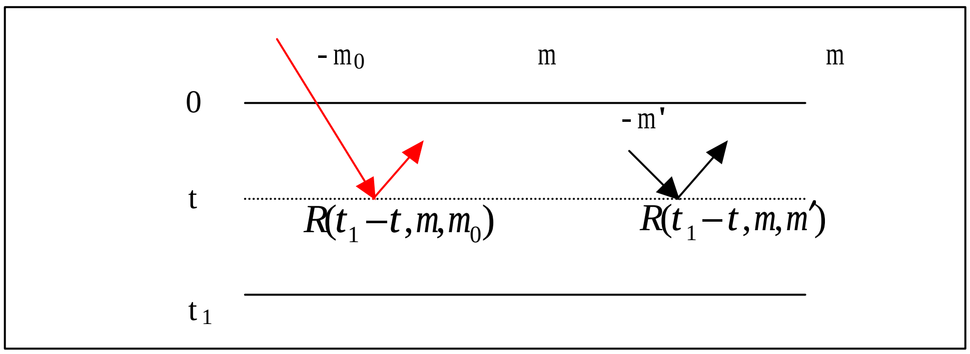
- **The principle of invariance for the <u>semi-infinite</u> atmosphere** (Ambartzumian, 1940): the diffuse reflected intensity cannot be changed if a layer of finite optical depth, having the same optical properties as those of the original layer, is added.

NOTE: For more details see L80: 6.4.2

- **The principles of invariance for the <u>finite</u> atmosphere** (Chandrasekhar, 1950):

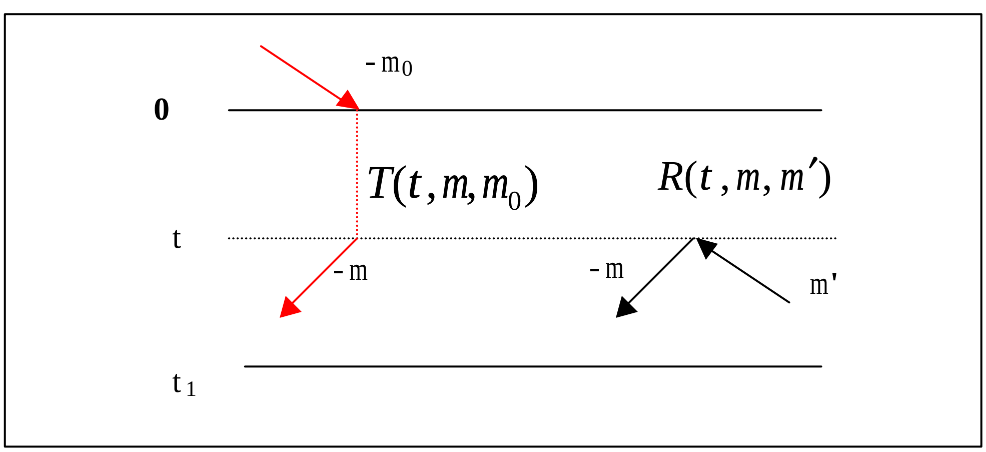
(1) The reflected (upward) intensity at any given optical depth t results from the reflection of (a) the attenuated solar flux = $pF_0 \exp(-t/m_0)$ and (b) the downward diffuse intensity at the level t. Thus, we have

$$I^{\uparrow}(t,m) = m_0 F_0 \exp(-t/m_0) R(t_1 - t, m, m_0) + 2\int_0^1 R(t_1 - t, m, m') I^{\downarrow}(t, -m') m' dm'$$ [21.1]



(2) The diffusely transmitted (downward) intensity at the level t results from (a) the transmission of incident solar flux and (b) the reflection of the upward diffuse intensity above the level t:

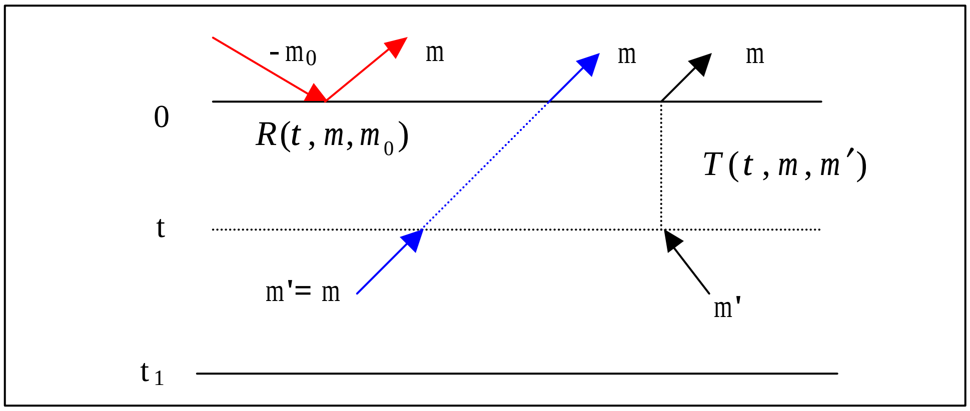
$$I^{\downarrow}(t,-m) = m_0 F_0 T(t, m, m_0) + 2\int_0^1 R(t, m, m') I^{\uparrow}(t, m') m' dm'$$ [21.2]



(3) The reflected (upward) intensity at the top of the finite atmosphere (t = 0) is equivalent to (a) the reflection of solar flux plus (b) the direct and diffuse transmission of the upward diffuse intensity above the level t:

$$I^{\uparrow}(0,m) = m_0 F_0 R(t,m,m_0) + 2\int_0^1 T(t,m,m')I^{\uparrow}(t,m')m'dm' + I^{\uparrow}(t,m)\exp(-t/m)$$

[21.3]

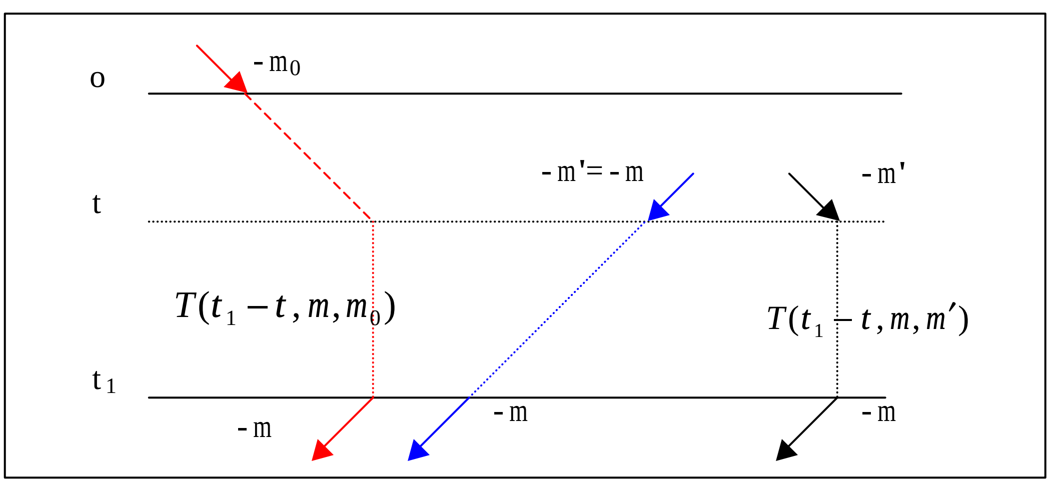


(4) The diffusely transmitted (downward) intensity at the bottom of the finite atmosphere (t = t1) is equivalent to (a) the transmission of the attenuated solar flux at the level t plus (b) the direct and diffuse transmission of the downward diffuse intensity at the level t from above:

$$I^{\downarrow}(t_1,-m) = m_0 F_0 \exp(-t/m_0) T(t_1 - t, m, m_0) + 2\int_0^1 T(t_1 - t, m, m')I^{\downarrow}(t,-m')m'dm' +$$

$$+ I^{\downarrow}(t,-m)\exp(-(t_1 - t)/m)$$

[21.4]



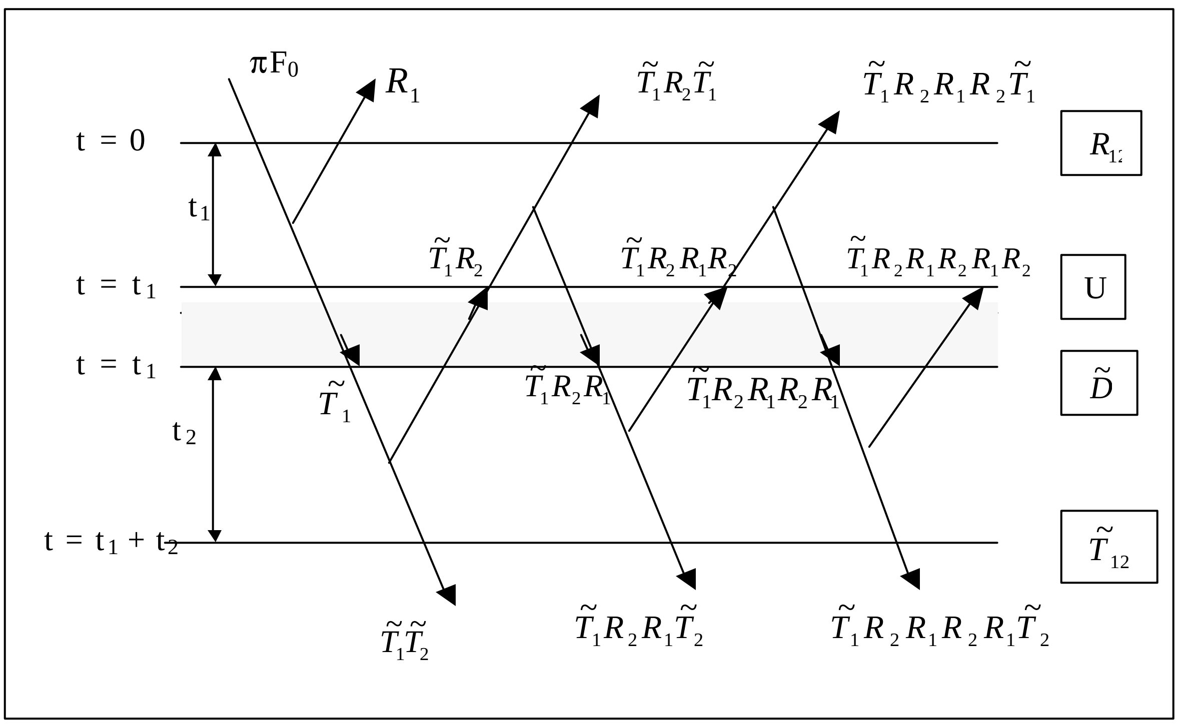
2. Adding method.

Adding method is an "exact" technique for solving the radiative transfer equation with multiple scattering. It uses geometrical ray-tracing approach and the reflection and transmission of each individual atmospheric layer.

Strategy: knowing the reflection and transmission of two individual layers, the reflection and transmission of the combined layer may be obtained by calculating the successive reflections and transmissions between these two layers.

NOTE: If optical depths of these two layers are equaled, this method is referred to as the doubling method.

Consider two layers with reflection R_1 and R_2 and total (direct plus diffuse) transmission $\tilde{T}_1$ and $\tilde{T}_2$ functions, respectively. Let's denote the combined reflection and total transmission functions by R_{12} and $\tilde{T}_{12}$, and combined reflection and total transmission functions between layers 1 and 2 by U and $\tilde{D}$, respectively.



The combined reflection function R_{12} is

$$R_{12} = R_1 + \tilde{T}_1 R_2 \tilde{T}_1 + \tilde{T}_1 R_2 R_1 R_2 \tilde{T}_1 + \tilde{T}_1 R_2 R_1 R_2 R_1 R_2 \tilde{T}_1 + \ldots =$$
$$= R_1 + \tilde{T}_1 R_2 \tilde{T}_1 [1 + R_1 R_2 + (R_1 R_2)^2 + \ldots] =$$
$$= R_1 + R_2 \tilde{T}_1^{\,2} (1 - R_1 R_2)^{-1} \qquad [21.5]$$

NOTE: In Eq.[21.5] we use that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

The combined total transmission function $\tilde{T}_{12}$ is

$$\tilde{T}_{12} = \tilde{T}_1 + \tilde{T}_1 R_2 R_1 \tilde{T}_2 + \tilde{T}_1 R_2 R_1 R_2 R_1 \tilde{T}_2 + \ldots =$$
$$= \tilde{T}_1 \tilde{T}_2 [1 + R_1 R_2 + (R_1 R_2)^2 + \ldots] =$$
$$= \tilde{T}_1 \tilde{T}_2 (1 - R_1 R_2)^{-1} \qquad [21.6]$$

The combined reflection function U between layers 1 and 2:

$$U = \tilde{T}_1 R_2 + \tilde{T}_1 R_2 R_1 R_2 + \tilde{T}_1 R_2 R_1 R_2 R_1 R_2 + \ldots =$$
$$= \tilde{T}_1 R_2 [1 + R_1 R_2 + (R_1 R_2)^2 + \ldots] =$$
$$= \tilde{T}_1 R_2 (1 - R_1 R_2)^{-1} \qquad [21.7]$$

The combined total transmission function $\tilde{D}$ between layers 1 and 2:

$$\tilde{D} = \tilde{T}_1 + \tilde{T}_1 R_2 R_1 + \tilde{T}_1 R_2 R_1 R_2 R_1 + \ldots =$$
$$= \tilde{T}_1 [1 + R_1 R_2 + (R_1 R_2)^2 + \ldots] =$$
$$= \tilde{T}_1 (1 - R_1 R_2)^{-1} \qquad [21.8]$$

From Eqs.[21.5]-[21.8], we find that

$$R_{12} = R_1 + \tilde{T}_1 U;\ \ \tilde{T}_{12} = \tilde{T}_2 \tilde{D};\ \ U = R_2 \tilde{D} \qquad [21.9]$$

Let's introduce $S = R_1 R_2 (1 - R_1 R_2)^{-1}$

Using that $\tilde{T} = T + \exp(-t / m')$, from Eqs.[21.8]-[21.9] we find

$$\begin{aligned} \tilde{D} &= D + \exp(-t_1 / m_0) = \\ &= (1+S)(T_1 + \exp(-t_1 / m_0)) = (1+S)T_1 + S\exp(-t_1 / m_0) + \exp(-t_1 / m_0) \end{aligned} \qquad [21.10]$$

$$\begin{aligned} \tilde{T}_{12} &= (T_2 + \exp(-t_2 / m_0))(D + \exp(-t_1 / m_0)) \\ &= D\exp(-t_2 / m_0) + T_2 \exp(-t_1 / m_0) + T_2 D + \exp\left(-\left[\frac{t_1}{m_0} + \frac{t_2}{m}\right]\right) d(m - m_0) \end{aligned} \qquad [21.11]$$

Thus, we may write a system of iterative equations for the computation of diffuse transmission and reflection for the two layers in the form:

$$\boxed{\begin{aligned} Q &= R_1 R_2 \\ S &= Q(1-Q)^{-1} \\ D &= T_1 + ST_1 + S\exp(-t_1 / m_0) \\ U &= R_2 D + R_2 \exp(-t_1 / m_0) \\ R_{12} &= R_1 + \exp(-t_1 / m)U + T_1 U \\ T_{12} &= \exp(-t_2 / m)D + T_2 \exp(-t_1 / m_0) + T_2 D \end{aligned}} \qquad [21.12]$$

NOTE: in Eq.[21.12], the product of two functions implies n integration over the appropriate angle so that all multiple-scattering contributions are included. For instance

$$R_1 R_2 = 2\int_0^1 R_1(m, m') R_2(m', m_0) m' dm'$$

Numerical procedure of the adding method:

1) As the starting point, one may calculate the reflection and transmission functions of an initial layer of very small optical depth (e.g., $\Delta\tau = 10^{-8}$) that the single scattering approximation is applicable.

NOTE: the solution of the radiative transfer under the single scattering approximation has been derived in Lecture 17 (see Eqs.[17.9]-[17.10])

2) Then, using Eq.[21.12], one computes the reflection and transmission functions of the layer of $2\Delta\tau$.

3) Using Eq.[21.12], one repeats the calculations adding the layers until a desirable optical depth is achieved.