

Lecture 3

Blackbody radiation. Main Laws. Simple aspects of radiative transfer.

Objectives:

1. Concepts of a blackbody, thermodynamical equilibrium, and local thermodynamical equilibrium.
2. Main laws:
 - Planck function.
 - Stefan-Boltzmann law.
 - Wien's displacement law.
 - Kirchhoff's law.
3. Simple aspects of radiative transfer:
 - Differential and integral forms of the radiative transfer equation
 - Formal solution of the radiative transfer in a plane-parallel atmosphere

Required reading:

L80: 1.2, 1.4

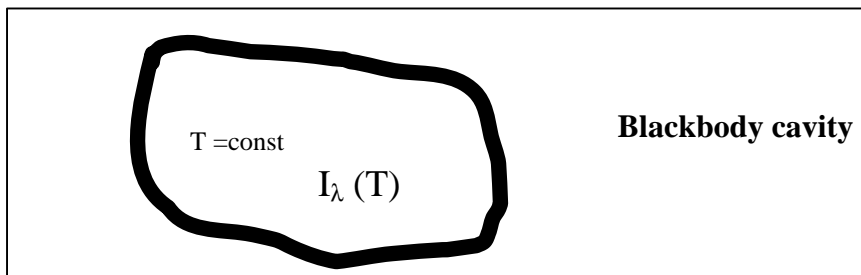
Additional/advanced reading:

L80: Appendix C; Le3.1-3.6/G&Y: 2.2.1-2.2.2

1. Concepts of a blackbody and thermodynamical equilibrium.

Thermodynamical equilibrium describes the state of matter and radiation inside an isolated constant-temperature enclosure.

Blackbody radiation is the radiative field inside a cavity in thermodynamic equilibrium.



NOTE: A blackbody cavity is an important element in the design of radiometers. Cavities are used to provide a well-defined source for calibration of radiometers. Another use of a cavity is to measure the radiation that flows into the cavity (e.g., to measure the radiation of sun). Further details will be provided in Lecture 9.

Properties of blackbody radiation:

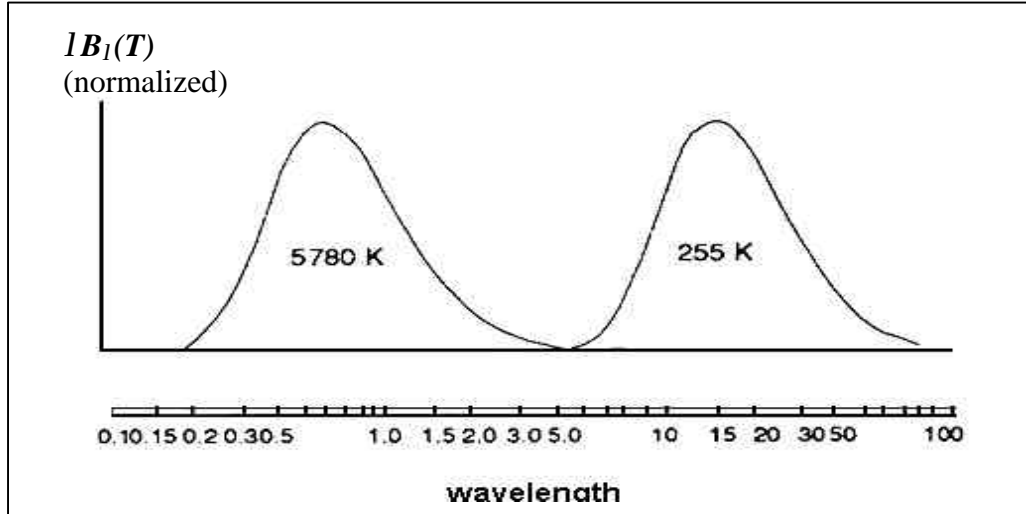
- Radiation emitted by a blackbody is isotropic, homogeneous and unpolarized;
- Blackbody radiation at a given wavelength depends only on the temperature T ;
- Any two blackbodies at the same temperature emit precisely the same radiation;
- A blackbody emits more radiation than any other type of an object at the same temperature;

NOTE: The atmosphere is not strictly in the thermodynamic equilibrium because its temperature and pressure are functions of position. Therefore, it is usually subdivided into small subsystems each of which is effectively isothermal and isobaric referred to as **Local Thermodynamical Equilibrium (LTE)**. (See G&Y89: 2.2.2 and Le93: 3.6.2 for more sophisticated explanation of a LTE concept).

- A concept of **LTE** plays a fundamental role in atmospheric studies: e.g., the main radiation laws discussed below, which are strictly speaking valid in **thermodynamical equilibrium**, can be applied to an atmospheric air parcel in **LTE**.

2. Main laws.

Planck function $B_l(T)$ gives the **intensity (or radiance)** emitted by a blackbody having a given temperature.



- **Planck function** can be expressed in wavelength, frequency, or wavenumber domains as

$$B_l(T) = \frac{2hc^2}{l^5 (\exp(hc / k_B T l) - 1)} \quad [3.1]$$

$$B_{\tilde{\nu}}(T) = \frac{2h\tilde{\nu}^3}{c^2 (\exp(h\tilde{\nu} / k_B T) - 1)} \quad [3.2]$$

$$B_n(T) = \frac{2hn^3 c^2}{\exp(hnc / k_B T) - 1} \quad [3.3]$$

where l is the wavelength; $\tilde{\nu}$ is the frequency; n is the wavenumber; h is the Planck's constant; k_B is the Boltzmann's constant ($k_B = 1.38 \times 10^{-23} \text{ J K}^{-1}$); c is the velocity of light; and T is the absolute temperature of a blackbody.

NOTE: The relations between $B_{\tilde{n}}(T)$; $B_n(T)$ and $B_l(T)$ are derived using that

$$B_{\tilde{n}}(T)d\tilde{n} = B_n(T)dn = B_l(T)dl, \text{ and that } l = c/\tilde{n} = 1/n$$

(See Lab 2 for more details).

NOTE: Because the sun and earth's spectra have a very small overlap, the radiative transfer processes for solar and infrared regions are often considered as two independent problems.

Asymptotic behavior of Planck function:

- If $l \rightarrow \infty$ (or $\tilde{n} \rightarrow 0$) (known as Rayleigh –Jeans distributions):

$$B_l(T) = \frac{2k_B T c}{l^4}$$

$$B_{\tilde{n}} = \frac{2k_B T \tilde{n}^2}{c^2}$$

NOTE: This longwave limit has a direct application to passive **microwave** remote sensing.

- If $l \rightarrow 0$ (or $\tilde{n} \rightarrow \infty$):

$$B_l(T) = \frac{2hc^2}{l^5} \exp(-hc / lk_B T)$$

$$B_{\tilde{n}} = \frac{2h\tilde{n}^3}{c^2} \exp(-h\tilde{n} / k_B T)$$

Stefan-Boltzmann law.

The **Stefan-Boltzmann law** states that the total power (energy per unit time) emitted by a **blackbody**, per unit surface area of the **blackbody**, varies as the fourth power of the temperature.

$$F = \rho B(T) = s_b T^4$$

where s_b is the *Stefan-Boltzmann constant* ($s_b = 5.671 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}$),

F is energy flux [W m^{-2}], and T is blackbody temperature [K];

$$B(T) = \int_0^{\infty} B_\lambda(T) d\lambda$$

Wien's displacement law.

The **Wien's displacement law** states that the wavelength at which the blackbody emission spectrum is most intense varies inversely with the blackbody's temperature. The constant of proportionality is Wien's constant (2897 K μm):

$$\lambda_m = 2897 / T$$

where λ_m is the wavelength (in micrometers, μm) at which the peak emission intensity occurs, and T is the temperature of the blackbody (in degrees Kelvin, K).

NOTE: this law is simply derived from $dB_\lambda/d\lambda = 0$

NOTE: Easy to remember statement of the Wien's displacement law:

the hotter the object the shorter the wavelengths of the maximum intensity emitted

Kirchhoff's law.

The **Kirchhoff's law** states that the emissivity, e_1 , of a medium is equal to the absorptivity, A_1 , of this medium under thermodynamic equilibrium:

$$e_1 = A_1$$

where e_1 is defined as the ratio of the emitting intensity to the Planck function;

A_1 is defined as the ratio of the absorbed intensity to the Planck function.

For a **blackbody**: $e_1 = A_1 = 1$ For a **non-blackbody**: $e_1 = A_1 < 1$

For a **gray body**: $e = A < 1$ (i.e., no dependency on the wavelength)

NOTE: Kirchhoff's law applies to gases, liquids and solids if they in TE or LTE.

- For atmospheric radiation transfer applications, one needs to distinguish between the **emissivity of the surface** (e.g., various types of lands, ice, etc.) and the **emissivity of an atmospheric volume** (consisting of gases, aerosols, and/or clouds).

Emissivity of the surfaces:

❖ In general, emissivity depends on the direction of emission, surface temperature, wavelength and some physical properties of the surface (e.g., the refractive index).

❖ In thermal IR ($\lambda > 4 \mu\text{m}$), nearly all surfaces are efficient emitters with the emissivity > 0.8 and their emissivity does not depend on the direction. Therefore, the

intensity emitted from a unit surface at a given wavelength is $I_1 = e_1 B_1(T_s)$

Table 3.1 Emissivity of some surfaces in the IR region from 10-to12 μm .

Surface	Emissivity
Water	0.993-0.998
Ice	0.98
Green grass	0.975-0.986
Sand	0.949-0.962
Snow	0.969-0.997
Granite	0.898

Emissivity of the atmospheric volume:

Absorption and thermal emission of the atmosphere volume is isotropic.

Kirchhoff's law applied to volume thermal emission gives

$$j_{1,thermal} = b_{a,1} B_1(T)$$

where $b_{a,1}$ is the absorption coefficient of the atmospheric volume and

j_1 is the **thermal emission coefficient** which relates to **the source function J_1**

(introduced in Lecture 2) as $J_1 = (j_{1,thermal} + j_{1,scattering}) / b_{e,1}$

and $b_{e,1}$ is the extinction coefficient of the atmospheric volume.

NOTE: In general, in contrast to emission/absorption, the scattering by the atmospheric volume is not isotropic (will be discussed later starting with Lecture 13).

3. Simple aspects of radiative transfer.

Let's consider a small volume DV of infinitesimal length ds and area DA containing optically active matter. Using the Extinction law (see Lecture 2), the change (loss plus gain due to both the thermal emission and scattering) of intensity along a path ds is

$$dI_1 = -b_{e,1} I_1 ds + b_{e,1} J_1 ds$$

Dividing this equation by $b_{e,1} ds$, we find

$$\frac{dI_1}{b_{e,1} ds} = -I_1 + J_1$$

[3.4]

Eq. [3.4] is the **differential equation of radiative transfer** called **Schwarzchild's equation**.

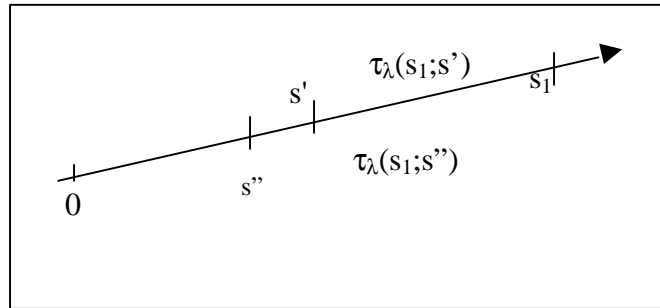
NOTE: Both I_1 and J_1 are generally functions of both position and direction.

As introduced in Lecture 2, the optical depth is

$$\mathbf{t}_I(s_1; s) = \int_s^{s_1} \mathbf{b}_{e,I}(s) ds$$

Thus

$$d\mathbf{t}_I = -\mathbf{b}_{e,I}(s) ds$$



Using the above expression for $d\tau_\lambda$, we can re-write Eq. [3.4] as

$$-\frac{dI_1}{dt_1} = -I_1 + J_1$$

or as

$$\frac{dI_1}{dt_1} = I_1 - J_1$$

[3.5]

These are other forms of the **differential equation of radiative transfer**.

Let's re-arrange terms in the above equation and multiply both sides by $\exp(-t_1)$. We have

$$-\frac{\exp(-t_1)dI_1}{dt_1} + \exp(-t_1)I_1 = \exp(-t_1)J_1$$

and (using that $d[I(x)\exp(-x)] = \exp(-x)dI(x) - \exp(-x)I(x)dx$) we find

$$-d[I_1 \exp(-t_1)] = \exp(-t_1)J_1 dt_1$$

Then integrating over the path from $\mathbf{0}$ to \mathbf{s}_1 we have

$$-\int_0^{s_1} d[I_1(s) \exp(-\mathbf{t}_1(s_1; s))] = \int_0^{s_1} \exp(-\mathbf{t}_1(s_1; s)) J_1 dt_1$$

and

$$-[I_1(s_1) - I_1(0) \exp(-\mathbf{t}_1(s_1; 0))] = \int_0^{s_1} \exp(-\mathbf{t}_1(s_1; s)) J_1 dt_1$$

Thus

$$I_1(s_1) = I_1(0) \exp(-\mathbf{t}_1(s_1; 0)) - \int_0^{s_1} \exp(-\mathbf{t}_1(s_1; s)) J_1 dt_1$$

and, using $d\mathbf{t}_1 = -\mathbf{b}_{e,1}(s) ds$, we have a **solution** of the **equation of radiative transfer** (often referred to as the **integral form of the radiative transfer equation**)

$$I_1(s_1) = I_1(0) \exp(-\mathbf{t}_1(s_1; 0)) + \int_0^{s_1} \exp(-\mathbf{t}_1(s_1; s)) J_1 \mathbf{b}_{e,1} ds \quad [3.6]$$

NOTE:

a) The above equation gives monochromatic intensity at a given point propagating in a given direction (often called an **elementary solution**). A completely general distribution of intensity in angle and wavelengths (or frequencies) can be obtained by repeating the elementary solution for all incident beams and for all wavelengths (or frequencies).

b) Knowledge of the source function J_1 is required to solve the above equation. In the general case, the source function consists of thermal emission and scattering (or emission from scattering), depends on the position and direction, and is very complex. One may say that the radiative transfer equation is all about the source function.

NOTE: The source function due to scattering will be introduced in Lecture 17.

- For a **non-scattering medium in the thermodynamical equilibrium**:

$$J_I = B_I(T), \text{ where } B_I(T) \text{ is Planck's function.}$$

Also, for non-scattering media, we have $\mathbf{b}_{e,I} = \mathbf{b}_{a,I} = k_I \mathbf{r}$, where k_I is the mass absorption coefficient and \mathbf{r} is the density (recall Lecture 2).

Thus, a **solution** of the **equation radiative transfer** for this case is

[3.7]

$$I_I(s_1) = I_I(0) \exp(-\mathbf{t}_I(s_1; 0)) + \int_0^{s_1} \exp(-\mathbf{t}_I(s_1; s)) B_I(T(s)) k_I \mathbf{r} ds$$

NOTE: The optical depth in Eq.[3.7] is due to absorption only, so

$$\mathbf{t}_I(s_1; s) = \int_s^{s_1} \mathbf{b}_{e,I}(s) ds = \int_s^{s_1} k_I \mathbf{r} ds$$

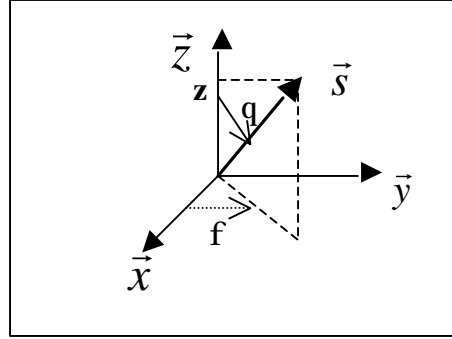
Plane-parallel atmosphere.

- For many applications, the atmosphere can be approximated by a **plane-parallel model** to handle the vertical stratification of the atmosphere.

Plane-parallel atmosphere consists of a certain number of atmospheric layers each characterized by homogeneous properties (e.g., T, P, optical properties of a given species, etc.) and bordered by the bottom and top infinite plates (called boundaries).

- Traditionally, the **vertical coordinate** z is used to measure linear distances in the plane-parallel atmosphere:

$$z = s \cos(\mathbf{q})$$



where q denotes the angle between the upward normal and the direction of propagation of a light beam (or zenith angle) and f is the azimuthal angle.

Therefore, using $ds = dz/\cos(q)$, the **radiative transfer equation** can be written as

$$\cos(\mathbf{q}) \frac{dI_1(z; \mathbf{q}; \mathbf{j})}{b_{e,1} dz} = -I_1(z; \mathbf{q}; \mathbf{j}) + J_1(z; \mathbf{q}; \mathbf{j})$$

Introducing the optical depth measured from the outer boundary downward as

$$t_1(z_1; z) = \int_z^{z_1} b_{e,1}(z) dz$$

and using $dt_1 = -b_{e,1}(z) dz$ and $m = \cos(q)$, we have

$$\boxed{m \frac{dI_1(t; m; \mathbf{j})}{dt} = I_1(t; m; \mathbf{j}) - J_1(t; m; \mathbf{j})} \quad [3.8]$$

Eq. [3.8] is the basic equation for the problem of radiative transfer in the plane-parallel atmosphere

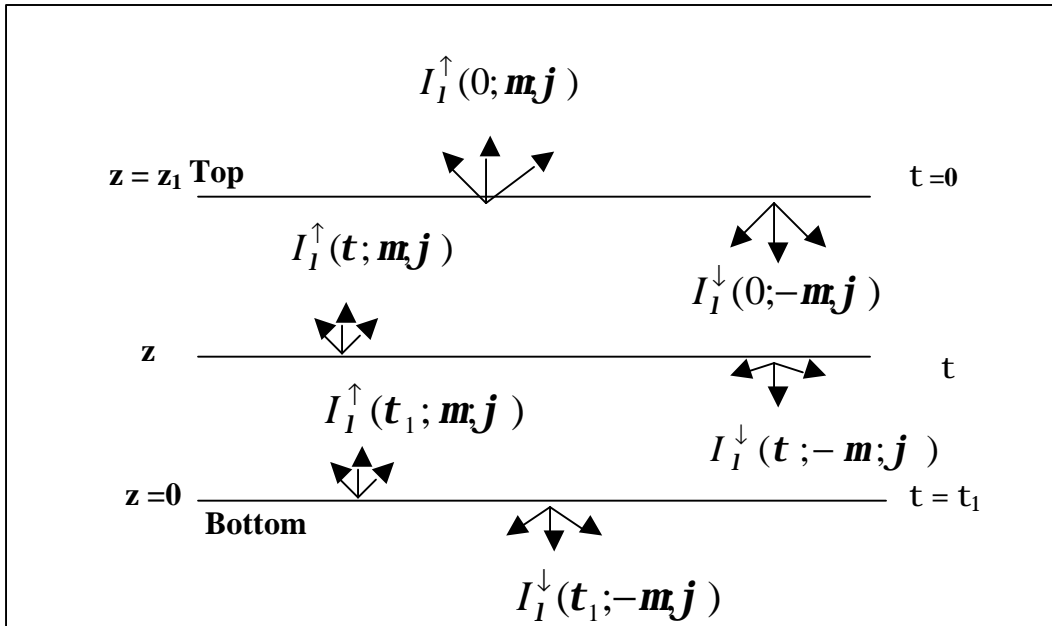
- Eq.[3.8] may be solve to give the **upward (or upwelling) and downward (or downwelling) intensities** for a finite atmosphere which is bounded on two sites.

Upward intensity I_1^\uparrow is for $1 \geq m \geq 0$ (or $0 \leq q \leq p/2$);

Downward intensity I_1^\downarrow is for $-1 \leq m \leq 0$ (or $p/2 \leq q \leq p$)

(using that $\cos(0)=1$; $\cos(p/2)=0$ and $\cos(p)=-1$)

Figure. Plane-parallel atmosphere



NOTE: For downward intensity, m is replaced by $-m$.

NOTE: The subindex l in t_1 was dropped to simplify notations.

The radiative transfer equation [3.8] can be written for upward and downward intensities:

$$m \frac{dI_1^\uparrow(t; \mathbf{m}; \mathbf{j})}{dt} = I_1^\uparrow(t; \mathbf{m}; \mathbf{j}) - J_1^\uparrow(t; \mathbf{m}; \mathbf{j}) \quad [3.9a]$$

$$-m \frac{dI_1^\downarrow(t; -\mathbf{m}; \mathbf{j})}{dt} = I_1^\downarrow(t; -\mathbf{m}; \mathbf{j}) - J_1^\downarrow(t; -\mathbf{m}; \mathbf{j}) \quad [3.9b]$$

These equations can be solved (following the procedure used to solve Eq.[3.4]) giving a solution for the upward and downward radiances.

A solution of Eq.[3.9a] gives a upward intensity in the plane-parallel atmosphere:

$$I_1^\uparrow(t; \mathbf{m}; \mathbf{j}) = I_1^\uparrow(t_1; \mathbf{m}; \mathbf{j}) \exp\left(-\frac{t_1 - t}{m}\right) + \frac{1}{m} \int_t^{t_1} \exp\left(-\frac{t' - t}{m}\right) J_1^\uparrow(t'; \mathbf{m}; \mathbf{j}) dt' \quad [3.10a]$$

and **a solution of Eq.[3.9b] gives a downward intensity in the plane-parallel atmosphere:**

$$I_1^\downarrow(t; -\mathbf{m}; \mathbf{j}) = I_1^\downarrow(0; -\mathbf{m}; \mathbf{j}) \exp\left(-\frac{t}{m}\right) + \frac{1}{m} \int_0^t \exp\left(-\frac{t - t'}{m}\right) J_1^\downarrow(t'; -\mathbf{m}; \mathbf{j}) dt' \quad [3.10b]$$

NOTE: Eq.[3.10a,b] for the **infrared radiative transfer** (the case of thermal absorption/emission) will be further discussed in Lecture 8, and for **solar radiative transfer** (the case of multiple scattering and absorption) will be discussed starting with Lecture 17.