Lecture 7

Terrestrial infrared radiative processes. Part 3:
Gaseous absorption/emission: Concept of an absorption band.
Absorption-band models. Curtis-Godson Approximation.

Objectives:

1. Concept of an absorption band. Absorption-band models:
   - Regular (Elsasser) band model
   - Statistical (Goody) band model
2. Curtis-Godson Approximation.

Required reading:

L80: 4.5; 4.6

Additional/Advanced reading:

Le93:9.3-9.6; 10/ G&Y89: 4.5; 4.6

1. Concept of an absorption band. Absorption-band models.

Band is a spectral interval of a width $\Delta\nu$ which is small enough to utilize a mean value of the Plank function $B_{\bar{\nu}}(T)$, but large enough so it consists of several **absorption lines**.

- **Absorption band models** are introduced to simplify the computation of the spectral transmittance (recall Lecture 6).

First, let's consider a homogeneous atmospheric layer (i.e., the spectral absorption coefficient $\boldsymbol{k_\nu}$ does not depend on path length).

Recall Lecture 6 where we have defined the **spectral transmission function** for a band of a width $\Delta\nu$ as

$$T_{\bar{\nu}}(u) = \frac{1}{\Delta\nu} \int_{\Delta\nu} \exp(-k_\nu u) d\nu = \frac{1}{\Delta\nu} \int_{\Delta\nu} \exp(-Sf(\nu - \nu_0)u) d\nu$$

Spectral absorptance can be defined as

$$\boxed{A_{\bar{n}}(u) = 1 - T_{\bar{n}}(u)}$$

Thus

$$A_{\bar{n}}(u) = 1 - T_{\bar{n}}(u) = \frac{1}{\Delta n}\int_{\Delta n}(1 - \exp(-k_n u))dn$$

$A_{\bar{n}}(u)\Delta n$ is called the **equivalent width *W(u)*.**

Equivalent width is

$$W(u) = A_{\bar{n}}\Delta n = \int_{-\infty}^{\infty}[1 - \exp(-k_n u)]dn$$

where W is in units of wavenumber.

Let's consider a band with **a single isolated line** of the **Lorentz profile**.

Using $k_\nu = S\, f(\nu - \nu_0))$ and the profile of Lorentz line, we have

$$A_{\bar{n}}(u) = \frac{1}{\Delta n}\int_{\Delta n}(1 - \exp\left(-\frac{S a u}{p\,(n - n_0)^2 + a^2}\right))dn$$

This integral can be expressed in term of the Ladendurg and Reiche function, L(x), as

$$\boxed{W = A_{\bar{n}}\Delta n = 2pa\,L(x)}$$

where $\boldsymbol{x = Su/2\pi\alpha}$

NOTE: The Ladendurg and Reiche function L(x) is given by the modified Bessel functions of the first kind of order n: $L(x) = I_0(x) + I_1(x)$, where $I_n(x) = i^{-n} J_n(ix)$ and

$$J_n(x) = \frac{i^{-n}}{p} \int_0^p \cos(nq) \;\; \exp(ix \;\; \cos(nq)) dq$$

For small x, L(x) is linear with its asymptotic expansion: L(x)= x[1-…]

For large x, L(x) is proportional to a square root of x: L(x) = $(2x/\pi)^{1/2}$[1-…]

Case of weak line absorption: either k_n or u is small => $\boldsymbol{k_n u} \ll \mathbf{1}$

Using the asymptotic of L(x) for small x, we have

$$A_{\bar{n}}(u) = \frac{W}{\Delta n} = 2pa\, L(x) / \Delta n \; = 2pa \; \frac{Su}{2pa\, \Delta n} = \frac{Su}{\Delta n}$$

Thus

$$\boxed{A_{\bar{n}}(u) = \frac{Su}{\Delta n}}$$

is called Linear absorption law.

Case of strong line absorption: $Su/pa \gg 1$

Using the asymptotic of L(x) for large x, we have

$$A_{\bar{n}}(u) = \frac{W}{\Delta n} = 2pa\, L(x) / \Delta n \; = 2pa \sqrt{\frac{2x}{p}} \; / \; \Delta n \; =$$

$$= 2pa \sqrt{\frac{2\,Su}{p\; 2pa}} \; / \; \Delta n \; = 2\sqrt{Su\; a} \; / \; \Delta n$$

Thus

$$\boxed{A_{\bar{n}}(u) = 2\,\frac{\sqrt{Su\; a}}{\Delta n}}$$

is called Square root absorption law.

- The weak and strong line absorption limits derived above are very helpful in developing the approximation of IR radiative transfer.

If lines in the various bands are uncorrelated, the multiplication law (see Lecture 6) works for average transmittance:

$$T_{\bar{n},1,2} = T_{\bar{n},1} T_{\bar{n},2}$$

Let's consider a band with several lines. Two cases can be identified: 1) lines have **regular positions** and 2) lines have **random positions**.

Regular Elsasser band model consists of an infinite array of Lorentz lines of equal intensity, spaced at equal intervals:

- This type of bands is similar to P and Q branches of linear molecules. For example, the spectrum of N_2O in 7.78 μm band; the spectrum of CO_2 in 15 μm band.

The **absorption coefficient** of the Elsasser bands with the line spacing δ is

$$k_n = \sum_{n=-\infty}^{\infty} \frac{S}{p} \frac{a}{(n - nd)^2 + a^2}$$

where δ is the line spacing (i.e., the distance in wavenumber domain (cm^{-1}) between the centers of two nearest lines).

Integrating the above expression for k_ν, it can be shown that the **spectral absorption** of the Elsasser band model is given as:

$$A_{\bar{n}} = erf\left(\frac{\sqrt{pSau}}{d} \right)$$

where $erf(x) = \frac{2}{\sqrt{p}} \int_0^x \exp(-x^2) dx$

Principle of statistical (random) models:

Many spectral bands have random line positions. To approximate this type of bands, various statistical models have been developed.

- H_2O 6.3 μm vibrational-rotational band and H_2O rotational band are characterized by random line positions.

Statistical (Goody) band model:

Randomly distributed Lorentz lines

Assuming that the probability distribution of intensities is the Poisson distribution

$$p(S) = \bar{S}^{-1} \exp(-S / \bar{S})$$

it can be shown (see the textbook) that the spectral transmittance can be expressed as

$$T_{\bar{n}}(u) = \exp\left(-\frac{\bar{S}u}{d} \left(1 + \frac{\bar{S}u}{pa} \right)^{1/2} \right)$$

where the $\bar{S}$ in the mean intensity and δ is the mean line spacing.

Malkmus model:

assumes that the probability distribution of intensities is

$$p(S) = \frac{1}{\ln(r)} \frac{1}{S} [\exp(-S / S_{m)} - \exp(-S \ r / S_m)]$$

where S_m is the maximum value of S and r is the ratio of maximum and minimum values of S.

For this case the spectral transmittance can be expressed as

$$T_{\bar{n}}(u) = \exp(-c_{\bar{n}}[(1 + d_{\bar{n}}u)^{1/2} - 1])$$

where

$$c_{\bar{n}} = 2\left(\sum_i \sqrt{a_i S_i} \right)^2 / \left(\Delta n \sum_i S_i \right); \quad d_{\bar{n}} = \left(\sum_i S_i \right)^2 / \left(\sum_i \sqrt{a_i S_i} \right)^2$$

2. Curtis-Godson Approximation

All discussion above was for homogeneous path. In real atmosphere of varying T and P some adjustments of the band models are needed to account for **inhomogeneous path** when

$$t = \int_u k_n(p,T)\,du$$

Strategy: reduce the radiative transfer problem to that of homogeneous path with some sort of averaged values of u*, T* and p*, so that optical depth can be computed accurately.

One-parameter scaling approximation:

Consider $\boldsymbol{T_r}$ and $\boldsymbol{p_r}$ are constant reference temperature and pressure and u* is variable scaled amount of absorber (scaled path length) such that

$$u^* = \int_u \left(\frac{p}{p_r}\right) R_{\bar{n}}(T, T_r, p_r)\,du$$

where

$$R_{\bar{n}}(T, T_r, p_r) = \frac{1}{\Delta n}\int_{\Delta n} R_n(T, T_r, p_r)\,dn$$

and for the Lorentz line

$$R_n(T, T_r, p_r) = \left(\frac{T_r}{T}\right)^{1/2} \sum_i \frac{S_i(T)\,a_i(p_r, T_r)}{(n - n_{o,i})^2} \Big/ \sum_i \frac{S_i(T_r)\,a(p_r, T_r)}{(n - n_{o,i})^2}$$

Therefore

$$\boxed{t_n = k_n(p_r, T_r)\,u^*}$$

Two-parameter scaling approximation (or Curtis-Godson approximation): introduces scaled mean intensity and line half-width (two adjustable parameters) as

$$\tilde{\bar{S}} = \int_0^u \bar{S}(T)du/u$$

$$\tilde{a} = \int_0^u \bar{S}(T)a(p,T)du / \int_0^u \bar{S}(T)du$$

Therefore

$$t = \int_u k_n(p,T)du = k_n(p^*,T^*)u$$

Using the statistical band model introduced above, **the spectral transmittance** for two-parameter approximation can be expressed as

$$T_{\bar{n}}(u) = \exp[\left(-\frac{\bar{S}}{d}\right)_r u^* \left(\frac{1+(\bar{S}/d)_r u^*}{(pa/d)_r\, p^*/p_r}\right)^{-1/2}]$$

- The two-parameter approximation for pressure and temperature corrections in connection with random band model has been widely used in IR radiative transfer calculations.